HOLOMORPHIC COUPLINGS IN STRING THEORY

Jan Louis [a] [*] and Kristin Förger [b]

[a]Martin–Luther–Universität Halle–Wittenberg,
FB Physik, D-06099 Halle, Germany

[b]Sektion Physik, Universität München,
Theresienstr. 37, D-80333 München, Germany

In these lectures we review the properties of holomorphic couplings in the effective action of four-dimensional $N = 1$ and $N = 2$ closed string vacua. We briefly outline their role in establishing a duality among (classes of) different string vacua.

0. Introduction

Holomorphic couplings of the (four-dimensional) low energy effective action play an important role in string theory. This is largely due to the fact that supersymmetry protects them against quantum corrections. In other words, they obey a non-renormalization theorem and this property considerably simplifies their computation in string theory. Furthermore, a certain subset of the holomorphic couplings can be calculated exactly and not only in a weak coupling (perturbative) expansion. As a consequence such couplings have been used to support some of the conjectured dualities between seemingly different four-dimensional string vacua.

In these lectures we review some of the (older) perturbative computations and outline their relevance for string duality. In particular, lecture 1 recalls some basic facts about perturbative string theories. Lecture 2 is devoted to $N = 1$ vacua of the heterotic string while lecture 3 focusses on $N = 2$ vacua of both the heterotic and type II string. Finally, in lecture 4 we discuss the heterotic–type II duality. At various points we make contact with other lectures presented at this school by K. Intriligator, S. Kachru, W. Lerche, K.S. Narain, R. Plesser and J. Schwarz.

1. Perturbative String Theory

1.1. String Loop Expansion

In string theory the fundamental objects are one-dimensional strings which, as they move in time, sweep out a two-dimensional worldsheet Σ. This worldsheet is embedded in some higher dimensional target space which is identified with a Minkowskian space-time. Particles in this target space appear as (massless) eigenmodes of the string and their scattering amplitudes are generalized by appropriate scattering amplitudes of strings. Strings can be open or closed, oriented or unoriented but in these lectures we solely focus on closed oriented strings. (For an introduction to string theory see for example [1–4].)

String scattering amplitudes are built from the fundamental vertex depicted in figure 1 which represents the splitting of a string or the joining of two strings. (Time is running horizontally.)
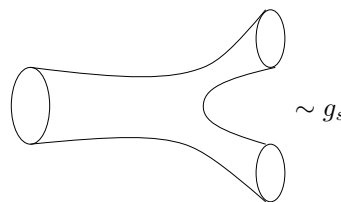

Figure 1. Fundamental string interaction

The strength of this interaction is governed by a dimensionless string coupling constant g_s. Out of the fundamental vertex one composes all other possible string scattering amplitudes, for example the four-point amplitude shown in figure 2.
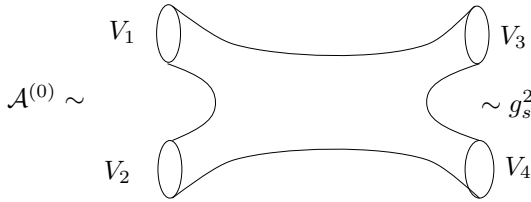

Figure 2. Four-point amplitude

The external 'tubes' should be thought of as extending into the far past and far future where the appropriate eigenstates of the string are prepared. Technically this is achieved by the string vertex operators V_i.

Obviously more complicated scattering processes – or equivalently more complicated worldsheets – involving a non-trivial topology can be built from the fundamental vertex. The Euclideanized version of any such worldsheet is a two-dimensional Riemann surface of a given genus n where n counts the number of holes in the worldsheet. The total N-point string scattering amplitude $\mathcal{A}$ is obtained by summing over all possible Riemann surfaces

$$\mathcal{A}(V_1, \ldots, V_N, g_s) = \sum_{n=0}^{\infty} \mathcal{A}^{(n)}(V_1, \ldots, V_N, g_s), \quad (1)$$

where $\mathcal{A}^{(n)}$ denotes the string scattering amplitude corresponding to a worldsheet of genus n. For example, a four-point amplitude of genus n together with its g_s dependence is displayed in figure 3.

For an arbitrary N-point amplitude the g_s dependence of $\mathcal{A}^{(n)}(V_1, \ldots, V_N, g_s)$ is easily found to be proportional to g_s^{2n+N-2} but one commonly absorbs one power of g_s into each vertex operator and defines $V_i' \equiv g_s V_i$. Using the rescaled vertex
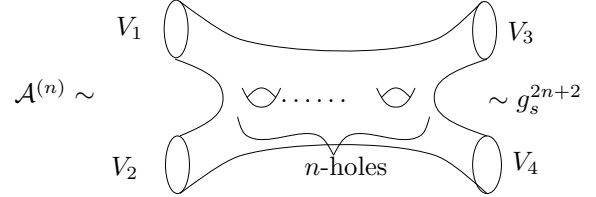

Figure 3. Riemann surfaces of genus n

operators V_i' one can eliminate the N dependence and define

$$\mathcal{A}^{(n)}(V_1', \ldots, V_N', g_s) = g_s^{2n-2} \mathcal{A}'^{(n)}(V_1', \ldots, V_N') \quad (2)$$

where $\mathcal{A}'^{(n)}(V_1', \ldots, V_N')$ no longer depends on g_s. As a consequence eq. (1) turns into

$$\mathcal{A}(V_1, \ldots, V_N, g_s) =$$
$$\sum_{n=0}^{\infty} g_s^{2n-2} \mathcal{A}'^{(n)}(V_1', \ldots, V_N') . \quad (3)$$

In this formula g_s appears with a power that coincides with the (negative of the) Euler number χ of the Riemann surface

$$\chi = \frac{1}{4\pi} \int_{\Sigma} \sqrt{h} R^{(2)} = 2 - 2n , \quad (4)$$

where h is the world-sheet metric and $R^{(2)}$ the two-dimensional curvature scalar.

From eq. (3) we learn that expanding $\mathcal{A}$ in powers of the string coupling g_s is equivalent to an expansion in worldsheet topologies. This expansion can also be interpreted as an expansion in the number of string loops and hence eq. (3) is also known as the string loop expansion. For $g_s < 1$ the scattering amplitude $\mathcal{A}^{(0)}$ which corresponds to a worldsheet of genus 0 or equivalently a sphere, is the dominant contribution while higher genus amplitudes are suppressed by higher powers of g_s. Our current understanding of string theory does not fix the strength of the string coupling and leaves g_s as a free parameter. The regime $g_s < 1$ then defines what is referred to as 'perturbative string theory'.

On the other hand, the strong coupling regime $g_s \geq 1$ was until recently inaccessible in that

there were no non-perturbative methods available for evaluating $\mathcal{A}$. However, during the past two years it was realized that the strong coupling region of a given string theory often can be mapped to another weakly coupled, 'dual' string theory and that most likely a non-perturbative formulation of string theory not only contains strings but also other extended objects of higher dimension. We briefly return to string dualities in lecture 4 but the recent developments have been nicely reviewed in the lectures by J. Schwarz and S. Kachru.

Despite the recent advances perturbative string theory has not gone out of fashion yet. As we will see in the course of these lectures the perturbative properties of the low energy effective actions and in particular their holomorphic couplings have played a vital part in supporting the validity of some of the conjectured dualities. Furthermore, the ultimate goal to connect string theory to physical phenomena at the weak scale requires a much more detailed knowledge about the perturbative sector as is currently available. Therefore, in this first lecture we briefly summarize some basic facts of string perturbation theory.

1.2. Conformal Field Theory

So far we merely isolated the g_s dependence of a string scattering amplitude but did not compute the heart of the matter $\mathcal{A}'^{(n)}(V_1', \ldots, V_N')$ in eq. (3). A detailed review of the rules and techniques for computing $\mathcal{A}'^{(n)}(V_1', \ldots, V_N')$ is beyond the scope of these lectures and we refer the reader to the literature for more details [1–7]. For our purpose we recall that the interactions of the string are governed by a two-dimensional field theory on the world-sheet Σ. $\mathcal{A}$ can be interpreted as an unitary scattering amplitude in the target space whenever the two-dimensional field theory is conformally invariant. The conformal group in two dimensions is generated by the infinite dimensional Virasoro algebra whose generators L_k obey

$$[L_k, L_j] = (k-j)\, L_{k+j} + \frac{c}{12}(k^3 - k)\, \delta_{k+j,0} \ . \quad (5)$$

The constant c is the central charge of the algebra which is constrained to vanish, i.e. $c = 0$. Since

string theories	supersymmetry	$(\bar{c}_{\mathrm{m}}, c_{\mathrm{m}})$
bosonic string	$(0,0)$	$(26, 26)$
superstring	$(1,1)$	$(15, 15)$
heterotic string	$(0,1)$	$(26, 15)$

Table 1
Worldsheet supersymmetry and central charges.

we are discussing oriented closed string theories the conformal field theory (CFT) is invariant under two separate conformal groups acting on the two light cone coordinates $\sigma \pm \tau$. (τ is the two-dimensional time and σ the space coordinate.) In fact the entire partition function splits into two sectors each of which carries a representation of the Virasoro algebra. These two sectors are commonly referred to as the left and right moving sector.

The different closed string theories are defined by the amount of local worldsheet supersymmetry. The bosonic string has no worldsheet supersymmetry, while the superstring has one supersymmetry in each the left and right moving sector; this is called $(1,1)$ supersymmetry. The heterotic string is a hybrid of the bosonic string and the superstring in that it has one supersymmetry only in the right moving sector or equivalently $(0,1)$ supersymmetry.

The central charge in eq. (5) has been normalized such that a free (two-dimensional) boson contributes $c = \bar{c} = 1$ and a (right moving) Majorana fermion has $c = \frac{1}{2}$ ($c(\bar{c})$ denotes the central charges of the right (left) moving sector). In addition to these 'matter fields' of the CFT also the reparametrization ghosts of the worldsheet contribute central charges $c_{\mathrm{g}}, \bar{c}_{\mathrm{g}}$. If there is no supersymmetry on the worldsheet one finds $c_{\mathrm{g}} = \bar{c}_{\mathrm{g}} = -26$ while a locally supersymmetric worldsheet has $c_{\mathrm{g}} = -15$. For the total central charge to vanish these ghost contributions have to be balanced by the central charge of the matter fields $c_{\mathrm{m}}, \bar{c}_{\mathrm{m}}$. For the closed string theories this situation is summarized in table 1.

The conformal symmetry ensures the consistency of the tree level scattering amplitude $\mathcal{A}^{(0)}$

but at higher loops an additional requirement has to be fulfilled. The two-dimensional field theory also has to be invariant under global reparametrizations of the higher genus Riemann surfaces. At genus 1 (torus) the group of global reparametrizations is the modular group $SL(2, \mathbf{Z})$ (some of its basic features are summarized in ref. [8] and appendix A). $SL(2, \mathbf{Z})$ invariance severely constrains the partition function of a CFT and thus the spectrum of physical states in the target space. In particular it automatically ensures an anomaly free effective field theory in the target space [9].

The bosonic string is afflicted with the problem of containing a (tachyonic) state with negative mass in its spectrum and the difficulty of constructing fermions in space-time. Therefore, we omit the bosonic string from our subsequent discussions and only focus on the superstring and the heterotic string. In both cases worldsheet supersymmetry requires the presence of two-dimensional fermions in the CFT. Such fermions can have different types of boundary conditions on the worldsheet: periodic (Ramond) or anti-periodic (Neveu-Schwarz). Modular invariance requires to sum over all possible boundary conditions of the worldsheet fermions and the states in the target space therefore arise in sectors with different fermion boundary conditions. For example, in the heterotic string the NS sector gives rise to space-time bosons while space-time fermions originate from the R sector. For the superstring there are worldsheet fermions in both the left and right moving sector so that there are altogether four distinct sectors $NS-NS$, $NS-R$, $R-NS$, $R-R$. Space-time bosons now arise from the $NS-NS$ or $R-R$ sector while space-time fermions appear in the $NS-R$ and $R-NS$ sector.

In order for a string to propagate in a d-dimensional target space (which should be identified with a Minkowskian space-time) a subset of the matter fields of the CFT have to be d free two-dimensional bosons together with the appropriate superpartners. These free fields build up what is called the 'space-time' (or universal) sector while the 'left over' fields can be an arbitrary (but modular invariant) interacting CFT called the 'internal' sector. The central charges of the two sec-

	$(\bar{c}_{\text{st}}, c_{\text{st}})$	$(\bar{c}_{\text{int}}, c_{\text{int}})$
superstring	$(\frac{3}{2}\,d, \frac{3}{2}\,d)$	$(15 - \frac{3}{2}\,d, 15 - \frac{3}{2}\,d)$
heterotic	$(d, \frac{3}{2}\,d)$	$(26 - d, 15 - \frac{3}{2}\,d)$

Table 2
Balance of central charges in d space-time dimensions.

	$NS - NS$	$R - R$
IIA	g_{MN}, b_{MN}, D	V_M, V_{MNP}
IIB	g_{MN}, b_{MN}, D	V^*_{MNPQ}, b'_{MN}, D'
Het	$g_{MN}, b_{MN}, D, V_M^{(a)}$	—

Table 3
Massless bosonic spectrum in $d = 10$.

tors are additive $c_{\text{m}} = c_{\text{st}} + c_{\text{int}}$, where $c_{\text{st}}(c_{\text{int}})$ is the central charge of the space-time (internal) sector. The balance of the central charges for a string propagating in a d-dimensional space-time is summarized in table 2.

The space-time sector containing free two-dimensional fields is more or less unique. However, the interacting internal CFT is only constrained by modular invariance and as we will see later also by the amount of space-time supersymmetry. However, in most cases one finds a whole plethora of CFT which satisfy all constrains. Each of these CFT together with their space-time sector is often referred to as a string vacuum.

The dimension d of space-time is completely arbitrary at this point. The simplest case is to choose as many free fields as possible which corresponds to $d = 10$ for both string theories. In this case the constraint from modular invariance is particularly strong and only leaves four consistent closed string theories: the non-chiral type IIA, the chiral type IIB and the heterotic string with a gauge group $E_8 \times E_8$ or $SO(32)$. Their massless bosonic spectrum is summarized in table 3.

In all four cases the $NS - NS$ sector contains the graviton g_{MN}, an antisymmetric tensor b_{MN} and a scalar D called the dilaton. The heterotic string also has gauge bosons $V_M^{(a)}$ in the adjoint representation of either $E_8 \times E_8$ or $SO(32)$. These are the two anomaly free gauge groups in ten dimensions and this choice is also dictated by modular invariance. The R-R sector of the type IIA string features an Abelian vector V_M and an antisymmetric 3-form V_{MNP}. For type IIB one finds an additional scalar D', a second antisymmetric tensor b'_{MN} and a self-dual antisymmetric 4-form V^*_{MNPQ}. The fermionic degrees of freedom are such that they complete the ten-dimensional supermultiplets. In type IIA one finds two spin-$\frac{3}{2}$ gravitini of opposite chirality (non-chiral $N = 2$), in type IIB there are two gravitini of the same chirality (chiral $N = 2$), and the heterotic string has one gravitino ($N = 1$) and one spin-$\frac{1}{2}$ gaugino also in the adjoint representation of $E_8 \times E_8$ or $SO(32)$.

The dilaton D plays a special role in string theory. Together with the antisymmetric tensor and the graviton it necessarily appears in all (perturbative) string theories. It is a flat direction of the effective potential so that its vacuum expectation value (VEV) $\langle D \rangle$ is a free parameter. More specifically, this VEV is directly related to the string coupling g_s via

$$\langle D \rangle = \ln g_s \ . \tag{6}$$

This can be seen on the one hand from the two-dimensional σ-model approach with an action [1, 10,11]:

$$S = S^*(g, b) + \frac{1}{4\pi} \int_\Sigma \sqrt{h} R^{(2)} D(x) \ . \tag{7}$$

If one expands the dilaton around its VEV $D = \langle D \rangle + \delta D$ and uses eq. (4) the action S shifts by the constant term $\delta S = (2 - 2n)\langle D \rangle$. This in turn generates a factor of $e^{(2n-2)\langle D \rangle}$ in the path integral or equivalently in all scattering amplitudes. Comparison with eq. (3) then leads to the identification (6). Alternatively one can derive (6) by explicitly calculating appropriate string scattering amplitudes [7].

1.3. Low Energy Effective Action

The space-time spectrum of a string theory contains a finite number of massless modes, which we denote as L, and an infinite number of massive modes H. Their mass is an (integer) multiple of the characteristic mass scale of string theory $M_{\rm str}$. Among the massless modes one always finds a spin-2 object which is identified with Einstein's graviton. This identification relates $M_{\rm str}$ to the characteristic scale of gravity $M_{\rm Pl}$. More specifically one finds [12,13]

$$M_{\rm str} \sim g_s^{1-d/2} M_{Pl} \tag{8}$$

up to a numerical constant which depends on the precise conventions chosen.

One is particularly interested in scattering processes of massless modes with external momentum p which is much smaller than $M_{\rm str}$, i.e. one wants to consider the limit $p^2/M_{\rm str}^2 \ll 1$. The aim is to derive a low energy effective action $\mathcal{L}_{\rm eff}(L)$ that only depends on the light modes L and where all heavy excitations H have been integrated out. This effective action can be reliably computed at energy scales far below $M_{\rm str}$. A systematic procedure for computing $\mathcal{L}_{\rm eff}(L)$ has been developed [6,7,14,15] and is often referred to as the S-matrix approach. One computes the S-matrix elements for a given string vacuum as a perturbative power series in g_s. At the lowest order (tree level) an S-matrix element typically has a pole in the external momentum which corresponds to the exchange of a massless mode L. The finite part is a power series in $p^2/M_{\rm str}^2$ and corresponds to the exchange of the whole tower of massive H-modes. $\mathcal{L}_{\rm eff}$ is then constructed to reproduce the string S-matrix elements in the limit $p^2/M_{\rm str}^2 \ll 1$ with S-matrix elements constructed entirely from the effective field theory of the L-modes. In this low energy effective theory the exchange of the H-modes in the string scattering is replaced by an effective interaction of the L-modes. For a four-point amplitude this procedure is schematically sketched in figure 4. The first row denotes the string scattering amplitude and its separation in a 'pole piece' (exchange of a massless mode) and the finite piece (exchange of the heavy modes). The second row indicates ordinary field-theoretical Feynman diagrams computed from the effective Lagrangian.
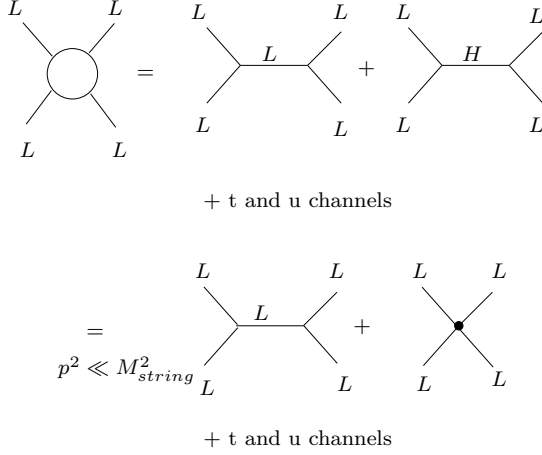
+ t and u channels

+ t and u channels

Figure 4. The S-matrix approach.

The pole piece is reproduced by the same exchange of the massless modes while the finite part is identified with an effective interaction. Using this procedure $\mathcal{L}_{\text{eff}}$ can be systematically constructed as a power series in both p^2/M_{str}^2 and g_s. The power of p^2 counts the number of space-time derivatives in $\mathcal{L}_{\text{eff}}$; at order $(p^2/M_{\text{str}}^2)^0$ one finds the effective potential while the order $(p^2/M_{\text{str}}^2)^2$ corresponds to the two-derivative kinetic terms.[2]

1.4. String Vacua in $d = 4$

For the rest of these lectures we concentrate on string vacua with four space-time dimensions. In this case the necessary central charges can be red off from table 2. For the superstring we have $(\bar{c}_{\text{st}}, c_{\text{st}}) = (6, 6)$ and $(\bar{c}_{\text{int}}, c_{\text{int}}) = (9, 9)$ while for the heterotic string $(\bar{c}_{\text{st}}, c_{\text{st}}) = (4, 6)$ and $(\bar{c}_{\text{int}}, c_{\text{int}}) = (22, 9)$ holds. The massless spectrum in the space-time or universal sector can be obtained by naive dimensional reduction from the $d = 10$ massless fields. Thus there always are a graviton $g_{mn}, m, n = 0, \ldots, 3$, an antisymmetric tensor b_{mn} and the dilaton D. In $d = 4$ the an-

tisymmetric tensor b_{mn} has one physical degree of freedom and is 'dual' to a Lorentz scalar a.[3] This duality can be made explicit through the field strength H_{npq} of the antisymmetric tensor

$$H^m := \frac{1}{3}\epsilon^{mnpq}H_{npq} = \epsilon^{mnpq}\partial_n b_{pq} \sim \partial^m a(x). \quad (9)$$

H^m is invariant under the local gauge transformation $\delta b_{mn} = \partial_m \xi_n(x) - \partial_n \xi_m(x)$ which transmogrifies into a continuous Peccei-Quinn (PQ) shift symmetry for the scalar $a(x)$

$$a(x) \to a(x) - \frac{\gamma}{4\pi} . \quad (10)$$

γ is an arbitrary real constant and the factor 4π has been introduced for later convenience. This PQ-symmetry holds to all orders in string perturbation theory but as we will see is generically broken by non-perturbative effects.

The internal sector of the heterotic string has central charges $(\bar{c}_{\text{int}}, c_{\text{int}}) = (22, 9)$. The left moving internal $\bar{c}_{\text{int}} = 22$ CFT together with the right moving $\bar{c}_{\text{st}} = 6$ CFT gives rise to (non-Abelian) gauge bosons of a gauge group G. In $d = 4$ or equivalently for a $\bar{c}_{\text{int}} = 22$ CFT the constraint from modular invariance is much less stringent as for $d = 10$ ($\bar{c}_{\text{int}} = 16$) and many gauge groups other than $E_8 \times E_8$ or $SO(32)$ are allowed. However, the size of G is not arbitrary but bounded by the central charge $\bar{c}_{\text{int}}$[4]

$$\text{rank}(G) \leq 28 . \quad (11)$$

The right moving $c_{int} = 9$ CFT can support space-time supercharges if it is invariant under additional (global) worldsheet supersymmetries.[5]

---

[2] Instead of using the S-matrix approach one can alternatively construct the effective action by computing the β-functions of the two-dimensional σ-model and interpreting them as the equations of motion of string theory. The effective action is then constructed to reproduce these equations of motion (for a review see for example [10,11,1]).

[3] This duality interchanges the Bianchi identity with the field equation of different tensors of the Lorentz group and has no relation with the string duality discussed earlier.

[4] The precise bound also depends on the number of space-time supersymmetries. For $N = 4$ one finds (11) while for $N = 2$ ($N = 1$) one has $\text{rank}(G) \leq 24$ ($\text{rank}(G) \leq 22$). Furthermore, all of these bounds only exists in perturbation theory. Non-perturbatively the gauge group can be enhanced beyond the bound imposed by the central charge [16]. We briefly return to this point in lecture 4.

[5] Strictly speaking there also is a condition on the (worldsheet) $U(1)$ charge of the primary states [17,18]. Alternatively, the conditions for space-time supersymmetry can be stated in terms of generalized Riemann identities of the partition function [19].

Heterotic String		
space-time SUSY	world-sheet SUSY	compact manifold
$N = 1$	$(0, 2)$	CY_3
$N = 2$	$(0, 4) \oplus (0, 2)$	$K3 \times T^2$
$N = 4$	$(0, 2) \oplus (0, 2) \oplus (0, 2)$	T^6

Table 4
Worldsheet and space-time supersymmetry of the heterotic string.

Superstring (left-right symmetric)		
space-time SUSY	world-sheet SUSY	compact manifold
$N = 1$	——	——
$N = 2$	$(2, 2)$	CY_3
$N = 4$	$(4, 4) \oplus (2, 2)$	$K_3 \times T^2$

Table 5
Worldsheet and space-time supersymmetry of the superstring.

For example, $N = 1$ space-time supersymmetry requires a (global) $(0, 2)$ supersymmetry of the $c_{int} = 9$ CFT [17]. In order to obtain $N = 2$ space-time supersymmetry one has to split the $c_{int} = 9$ CFT into a free $c_{int} = 3$ CFT with $(0, 2)$ worldsheet supersymmetry and a $c_{int} = 6$ CFT with $(0, 4)$ supersymmetry [18]. Finally, a heterotic vacuum with $N = 4$ space-time supersymmetry is constructed by splitting the $c_{int} = 9$ CFT into three $c_{int} = 3$ CFT each with $(0, 2)$ world-sheet supersymmetry.

The previous discussion related the amount of space-time supersymmetry to properties of the internal CFT in particular to the amount of worldsheet supersymmetry. A subset of these CFT can be associated with a compact manifold on which the ten-dimensional heterotic string is compactified. Such compact manifolds have to be six-dimensional Ricci-flat Kähler manifolds with a holonomy group contained in $SU(3)$; such manifolds are termed Calabi–Yau manifolds and we summarize some of their properties in appendix B. $N = 1$ is obtained when the holonomy group is exactly $SU(3)$ which corresponds to a Calabi-Yau threefolds CY_3. $N = 2$ requires a holonomy group $SU(2)$ corresponding to the two-dimensional $K3$ surface times a two-torus T^2. Finally a toroidal compactification of the heterotic string leaves all supercharges intact and thus has $N = 4$ supersymmetry. We summarize the conditions for space-time supersymmetry in table 4.

A similar discussion for the superstring depends on the symmetry between left and right moving sectors. In these lectures we only consider left-right symmetric type II vacua and without further discussion we summarize the relations between space-time supersymmetry, worldsheet supersymmetry and the compactification manifolds in table 5.

2. $N = 1$ Heterotic Vacua in $d = 4$

We start our discussion of the low energy effective action with the class of four-dimensional heterotic string vacua whose spectrum and interactions are $N = 1$ supersymmetric. These are also the string vacua which have mostly been studied so far because of their phenomenological prospects. Let us recall some basic facts of supersymmetry and supergravity following the notation and conventions of [20].

2.1. $N = 1$ Supergravity

$N = 1$ supersymmetry is generated by four fermionic charges Q_α and $Q_{\dot\alpha}$, which transform as Weyl spinors of opposite chirality under the Lorentz group. They obey the supersymmetry algebra

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2\sigma_{\alpha\dot\alpha}^m p_m \ , \tag{12}$$

where p_m is the four-momentum and σ^m are the Pauli matrices.

There are four distinct $N = 1$ supersymmetric multiplets in $d = 4$, the gravitational multiplet E, the vector multiplet V, the chiral multiplet Φ and finally the linear multiplet L. The gravitational multiplet consists of the graviton g_{mn} and gravitino $\psi_{m\alpha}$; the vector multiplet features a gauge boson v_m and a spin-$\frac{1}{2}$ gaugino

$N = 1$ multiplets		spin
gravity	$E \sim (g_{mn}, \psi_{m\alpha})$	$(2, \frac{3}{2})$
vector	$V \sim (v_m, \lambda_\alpha)$	$(1, \frac{1}{2})$
chiral	$\Phi \sim (\chi_\alpha, \phi)$	$(\frac{1}{2}, 0)$
linear	$L \sim (\chi_\alpha, H_m, l)$	$(\frac{1}{2}, 0)$

Table 6
$N = 1$ multiplets.

λ_α while the chiral multiplet contains a complex scalar ϕ and a chiral Weyl spinor χ_α. The linear multiplet contains a real scalar l, a Weyl fermion χ_α and a conserved vector H^m, which is the field strength of an antisymmetric tensor $H^m = \frac{1}{3}\epsilon^{mnpq} H_{npq} = \epsilon^{mnpq}\partial_n b_{pq}$. All four multiplets have two bosonic and two fermionic physical degrees of freedom and they are summarized in table 6.

As we already discussed in the last lecture an antisymmetric tensor in $d = 4$ is dual to a real scalar $a(x)$ (c.f. eq. (9)). In $N = 1$ this duality generalizes to a duality between an entire linear and a chiral multiplet [21–23]. In particular the complex scalar S of the dual chiral multiplet is given by $S = l + ia$ so that the continuous PQ-symmetry (10) acts on S according to

$$S \to S - \frac{i\gamma}{4\pi} \ . \tag{13}$$

This symmetry holds to all orders in perturbation theory and strongly constrains the possible interactions of the dual chiral multiplet. We choose to eliminate the linear multiplet from our subsequent discussions and express all couplings in terms of the dual chiral multiplet. This simplifies some of the formulas below but more importantly at the non-perturbative level the physics is more easily captured in the chiral formulation.[6] However, we should stress that some of the perturbative properties that we will encounter can be

---

[6]The recent progress about non-perturbative properties of string vacua indicate that the appearance of an antisymmetric tensor is an artifact of string perturbation theory and that in a non-perturbative formulation the dilaton sits in a chiral multiplet [24–28]. We return to this aspect in lecture 4.

understood on a more conceptual level by using the linear formulation [29–34].[7]

With this in mind let us recall the bosonic terms of the most general gauge invariant supergravity Lagrangian with only chiral and vector multiplets and no more than two derivatives [38,20]

$$
\begin{aligned}
\mathcal{L} = &-\frac{3}{\kappa^2}\int d^2\theta d^2\bar{\theta} E\, e^{-\frac{1}{3}\kappa^2 K(\Phi,\bar{\Phi},V)} \\
&+\frac{1}{4}\int d^2\theta \mathcal{E} \sum_a f_a(\Phi)(\mathcal{W}_\alpha \mathcal{W}^\alpha)_a + h.c. \\
&+\int d^2\theta \mathcal{E}\, W(\Phi) + h.c. \\
= &-\sqrt{g}\Big(\frac{1}{2\kappa^2}R + G_{I\bar{J}}\mathcal{D}_m\bar{\phi}^{\bar{J}}\mathcal{D}^m\phi^I + V(\phi,\bar{\phi}) \\
&+\sum_a \frac{1}{4g_a^2}(F_{mn}F^{mn})_a + \frac{\theta_a}{32\pi^2}(F\tilde{F})_a \\
&+\ \text{fermionic terms}\Big)
\end{aligned}
\tag{14}
$$

where $\kappa^2 = \frac{8\pi}{M_{Pl}^2}$; E is the superdeterminant and $\mathcal{E}$ the chiral density (for a precise definition of the superfield action, see [20]).

Supersymmetry imposes constraints on the couplings of $\mathcal{L}$ in eq. (14). The metric $G_{I\bar{J}}$ of the manifold spanned by the complex scalars ϕ^I is necessarily a Kähler metric and therefore obeys

$$G_{I\bar{J}} = \frac{\partial}{\partial \phi^I}\frac{\partial}{\partial \bar{\phi}^{\bar{J}}} K(\phi,\bar{\phi}) \ , \tag{15}$$

where $K(\phi,\bar{\phi})$ is the Kähler potential. It is an arbitrary real and gauge invariant function of ϕ and $\bar{\phi}$.

The gauge group G is in general a product of simple group factors G_a labelled by an index a, i.e.

$$G = \otimes_a G_a \ . \tag{16}$$

With each factor G_a there is an associated gauge coupling g_a which can depend on the ϕ^I. However, supersymmetry constrains the possible functional dependence and demands that the (inverse)

---

[7] For a discussion of field-theoretical non-perturbative effects (gaugino condensation) in the linear multiplet formalism see refs. [35–37].

gauge couplings g_a^{-2} are the real part of a holomorphic function $f_a(\phi)$ called the gauge kinetic functions. The imaginary part of the $f_a(\phi)$ are (field-dependent) θ-angles. One finds

$$\begin{aligned} g_a^{-2} &= \mathrm{Re}f_a(\phi) \ , \\ \theta_a &= -8\pi^2 \mathrm{Im}f_a(\phi) \ . \end{aligned} \qquad (17)$$

The scalar potential $V(\phi,\bar\phi)$ is also determined by a holomorphic function, the superpotential $W(\phi)$

$$V(\phi,\bar\phi) = e^{\kappa^2 K}\left(D_I W G^{I\bar J}\bar D_{\bar J}\bar W - 3\kappa^2 |W|^2\right), (18)$$

where $D_I W := \frac{\partial W}{\partial \phi^I} + \kappa^2 \frac{\partial K}{\partial \phi^I} W$.

To summarize, $\mathcal{L}$ is completely determined by three functions of the chiral multiplets, the real Kähler potential $K(\phi,\bar\phi)$, the holomorphic superpotential $W(\phi)$ and the holomorphic gauge kinetic functions $f_a(\phi)$.

However, there is a certain redundancy in this description. From eq. (15) we learn that the Kähler metric $G_{I\bar J}$ is invariant under a harmonic shift of the Kähler potential $K(\phi,\bar\phi) \to K(\phi,\bar\phi) + F(\phi) + \bar F(\bar\phi)$. The entire Lagrangian (14) shares this invariance if the superpotential is simultaneously rescaled while the gauge kinetic function is kept invariant. Altogether $\mathcal{L}$ is invariant under the replacements [8]

$$\begin{aligned} K(\phi,\bar\phi) &\to K(\phi,\bar\phi) + F(\phi) + \bar F(\bar\phi) \\ W(\phi) &\to W(\phi)\, e^{-F(\phi)} \\ f_a(\phi) &\to f_a(\phi) \ . \end{aligned} \qquad (19)$$

2.2. $N=1$ Heterotic String

Let us now turn to the heterotic string and determine some of the generic properties of K, W and f_a. In section 1.2. we briefly described a systematic procedure (the S-matrix approach) of how to compute the effective Lagrangian. Supersymmetry simplifies this project considerably since it reduced the arbitrary and hence unknown couplings to just K, W and f_a. In section 1.3. we already discussed the special role played by the

dilaton and its relation to the string coupling. From eqs. (3),(6) one also infers that there exists a particular coordinate frame – called the string frame – where the dilaton multiplies the entire tree level Lagrangian. In this frame the bosonic part of the effective action is given by

$$\begin{aligned} \mathcal{L}^{(0)} = \sqrt{\hat g}\,e^{-2D}\Big\{ &- \frac{1}{2\kappa^2}\hat R - \frac{1}{4}\sum_a k_a(F_{mn}F^{mn})_a \\ &- \frac{2}{\kappa^2}\partial_m D\partial^m D + \frac{1}{16\kappa^2}H_m H^m \\ &- \tilde G_{I\bar J}\mathcal{D}_m\phi^I \mathcal{D}^m\bar\phi^{\bar J} - V(\phi,\bar\phi)\Big\} \ , \end{aligned} \qquad (20)$$

where the ϕ^I now denote all massless scalar fields in the string spectrum except the dilaton D and axion a. The constant k_a is a positive integer (for non-Abelian gauge groups) and is the level of the left moving $\bar c = 22$ Kač-Moody algebra whose zero modes generate the space-time gauge bosons. H_m is a modified field strength which also contains Chern–Simons couplings of the antisymmetric tensor b_{mn} with the gauge fields and gravitons. From amplitudes like $\langle b_{mn}v_p v_q\rangle$ or $\langle b_{mn}g_{pr}g_{qs}\rangle$ one obtains [1,7,39,23,32]

$$H^m = \epsilon^{mnpq}\partial_n b_{pq} + \kappa^2(\omega_L^m - \sum_a k_a\,\omega_a^m) \ , \quad (21)$$

where ω_a^m is the Yang-Mills Chern-Simons term defined as $\omega_a^m = \epsilon^{mnpq}(v_n F_{pq} + \frac{2i}{3}v_n v_p v_q)_a$ and ω_L^m is the appropriate Lorentz Chern-Simons term.

In the string frame (20) the Einstein term is not canonically normalized and therefore the effective string Lagrangian cannot yet be compared with the supergravity Lagrangian (14). However, a Weyl rescaling of the space time metric

$$\hat g_{mn} = e^{2D}g_{mn} \qquad (22)$$

in eq. (20) results in a canonical Einstein term. In addition, one has to perform the duality transformation of the antisymmetric tensor[9] and then combine the dilaton D and the axion a into a complex scalar field

$$S = e^{-2D} + ia \ . \qquad (23)$$

---

[8] A nowhere vanishing W can be completely absorbed by a Kähler transformation and one defines $G := K + \ln|W|^2$ [38]. However, W usually does have zeros and it is necessary to keep a separate K and W.

[9] One first treats H^m as an unconstrained vector and imposes the constraint $\partial_m H^m = \kappa^2(R\tilde R - \sum_a k_a(F\tilde F)_a)$ with a Lagrange multiplier $a(x)$. Then the variation with respect to H^m implies $H_m \sim e^{4D}\partial_m a(x)$. (For more details see for example refs. [32,33].)

After these manipulations one arrives at

$$\mathcal{L}^{(0)} = -\sqrt{g}\Big\{\frac{1}{2\kappa^2}R \; + \; \tilde{G}_{I\bar{J}}\,\mathcal{D}_m\phi^I\mathcal{D}^m\bar{\phi}^{\bar{J}}$$
$$+ \; G_{S\bar{S}}\,\partial_m S\,\partial^m\bar{S} + \frac{1}{\mathrm{Re}S}\,V(\phi,\bar{\phi}) \qquad (24)$$
$$+ \; \sum_a \frac{k_a}{4}\left(\mathrm{Re}S\,(F_{mn}F^{mn})_a - \mathrm{Im}S\,(F\tilde{F})_a\right)\Big\},$$

where $G_{S\bar{S}} = \frac{1}{\kappa^2(S+\bar{S})^2} = \partial_S\partial_{\bar{S}}K$. Now one can easily do the comparison with the supergravity Lagrangian (14) and determine

$$\begin{aligned} K^{(0)} &= -\kappa^{-2}\ln(S+\bar{S}) + \tilde{K}^{(0)}(\phi,\bar{\phi}) \\ W^{(0)} &= W(\phi) \text{ i.e. } \partial_S W^{(0)} = 0 \qquad (25) \\ f_a^{(0)} &= k_a S\,, \end{aligned}$$

where $\tilde{G}_{I\bar{J}} = \partial_I\partial_{\bar{J}}\tilde{K}$. Note that the PQ-symmetry (13) shifts the Lagrangian (24) by a total divergence and thus the perturbative action is indeed invariant.[10]

2.2.1. Perturbative Corrections

So far the analysis was confined to the string tree level. The next step is to include string loop corrections into the effective Lagrangian. In section 1.2. we already determined the relation between the dilaton and the string coupling constant which organizes the string loop expansion. In fact, eqs. (3),(6) show that the dilaton dependence of a given coupling on a genus n Riemann surface is fixed. Therefore all higher loop corrections of the functions K, f and W are constrained by the following two properties:

- $e^{2D} = \frac{2}{S+\bar{S}}$ organizes the string perturbation theory with large S corresponding to weak coupling,

- the PQ-symmetry (13) is unbroken in perturbation theory.

Using these constraints one expands K, f_a and W in powers of the dilaton in accord with the string loop expansion. For the Kähler potential one finds

$$K = K^{(0)} + \sum_{n=1}^{\infty}\frac{\tilde{K}^{(n)}(\phi,\bar{\phi})}{(S+\bar{S})^n}\,, \qquad (26)$$

---

[10] Here we only considered the bosonic part of the Lagrangian but the analysis can be extended to the entire Lagrangian.

where the $\tilde{K}^{(n)}$ are arbitrary functions of the scalar fields ϕ^I but do not depend on the dilaton. The superpotential W and the gauge kinetic function f_a are additionally constrained by their holomorphy. Since $W^{(0)}$ does not depend on the dilaton, the only possible loop corrections (which are also invariant under the PQ-symmetry) could look like $W^{(n)}(S,\phi) \sim \frac{W^{(n)}(\phi)}{(S+\bar{S})^n}$ for $n > 1$, but any such term is incompatible with the holomorphy of W. Therefore W cannot receive corrections in string perturbation theory. This is a special case of the non-renormalization theorem of the superpotential which holds in any $N=1$ supersymmetric field theory [40]. The 'stringy' proof of this theorem which we recalled above was first presented in ref. [41].

The exact same considerations also imply a non-renormalization theorem for the gauge kinetic functions f_a [42,43]. One finds that only a dilaton independent one-loop correction $f_a^{(1)}(\phi)$ can arise in perturbation theory and hence

$$f_a = k_a S + f_a^{(1)}(\phi)\,. \qquad (27)$$

2.2.2. Non-Perturbative Corrections

Non-perturbative corrections to the couplings of the effective Lagrangian generically break the continuous PQ-symmetry to its anomaly free discrete subgroup. Space-time instantons generate a non-trivial topological density $\frac{1}{32\pi^2}\int F\tilde{F}$ and therefore break the PQ-symmetry to

$$S \to S - \frac{in}{4\pi}\,, \qquad (28)$$

where n is an integer and no longer a continuous parameter. The holomorphic invariants of (28) include the exponential $e^{-8\pi^2 S}$ and thus beyond perturbation theory one expects non-perturbative corrections of W and f to have the form:

$$\begin{aligned} W &= W^{(0)}(\phi) + W^{(NP)}(e^{-8\pi^2 S},\phi)\,, \\ f_a &= k_a S + f_a^{(1)}(\phi) + f_a^{(NP)}(e^{-8\pi^2 S},\phi)\,. \quad (29) \end{aligned}$$

To summarize, we learned in this section that the dilaton dependence of the couplings K, W and f_a is fixed. The dependence on all other scalar fields ϕ^I cannot be determined in general; it depends on the particular string vacuum under consideration or in other words the details

of the internal CFT. Furthermore, the quantum corrections of the holomorphic W and f_a are strongly constrained by non-renormalization theorems. (The non-renormalization theorems are also discussed from a slightly different point of view in lectures by K. Intriligator.)

2.3. Supersymmetric Gauge Couplings
2.3.1. Preliminaries

Up to now we denoted all scalars except the dilaton by ϕ^I. Let us introduce a further distinction and separate the scalars into matter fields Q^I that are charged under the gauge group and gauge neutral scalar fields M^i, called moduli. The moduli are flat directions of the effective potential in that they satisfy $\frac{\partial V}{\partial M^i} = 0$ for arbitrary $\langle M^i \rangle$. Hence, the VEVs $\langle M^i \rangle$ are free parameters of the string vacuum and they can be viewed as the coordinates of a (multi-dimensional) parameter space called the moduli space. On the other hand the vacuum expectation values of the Q^I are fixed by the potential.[11] One conveniently chooses $\langle Q^I \rangle = 0$ and expands all couplings around this expectation value. In particular we need

$$
\begin{aligned}
\tilde{K}(\phi, \bar{\phi}) &= \kappa^{-2} \, \hat{K}(M, \bar{M}) \\
&\quad + Z_{\bar{I}J}(M, \bar{M}) \, \bar{Q}^{\bar{I}} e^{2V} Q^J + \dots \\
f^{(1)}(\phi) &= f_a(M) + \dots \;,
\end{aligned} \tag{30}
$$

where the ellipsis stand for higher order terms that are irrelevant for our purpose. The couplings $\hat{K}(M, \bar{M})$, $Z_{\bar{I}J}(M, \bar{M})$ and $f_a(M)$ do not depend

on the dilaton but only on the moduli; in general this functional dependence cannot be further specified.

2.3.2. Field Theory Considerations

Any effective field theory has two distinct kinds of gauge couplings, a momentum dependent (running) effective gauge coupling $g_a(p^2)$, and a Wilsonian gauge coupling. Shifman and Vainshtein [42] stressed the importance of this distinction for supersymmetric field theories. It arises from the fact that the Wilsonian gauge coupling is the real part of a holomorphic function $\mathrm{Re} f_a$ which is not renormalized beyond one-loop. This Wilsonian coupling is the gauge coupling of a Wilsonian effective action which is defined by only integrating out the heavy and high frequency modes with momenta above a given cutoff scale. By construction such a Wilsonian Lagrangian is local and its couplings obey the analytic and renormalization properties discussed in the previous section.[12] On the other hand $g_a(p^2)$ are the couplings in the one-particle irreducible generating functional which includes momenta at all scales; it is related to physical quantities such as scattering amplitudes. At the tree level the two couplings coincide and we have $(g_a^{(0)})^{-2} = \mathrm{Re} f_a^{(0)}$. However, at the one-loop level they start to disagree and for any locally supersymmetric field theory one finds instead [42,46,29–32,44,47]

$$
\begin{aligned}
g_a^{-2}(p^2) &= \mathrm{Re} f_a + \frac{b_a}{16\pi^2} \ln \frac{\Lambda^2}{p^2} + \frac{c_a}{16\pi^2} \hat{K}^{(0)} \\
&\quad + \frac{T(G_a)}{8\pi^2} \ln g_a^{(0)-2} - \sum_r \frac{T_a(r)}{8\pi^2} \ln \det Z_r^{(0)} \;,
\end{aligned} \tag{31}
$$

where r runs over the representation of the gauge group and Λ is a (moduli independent) UV cutoff of the regularized supersymmetric quantum field theory which is naturally chosen as the Planck mass $\Lambda = M_{Pl}$.[13] $\hat{K}^{(0)}$ and $Z_r^{(0)}$ are the tree level couplings of the light (or massless) modes and b_a is the one-loop coefficient of their β-function.[14]

---

[11] There also can be gauge neutral singlets which are not moduli in that there VEV is fixed by the potential. Such fields are included among the Q^I. Furthermore, there often are also charged states which are flat directions of V. Their VEVs break the gauge group and give the associated gauge bosons a mass. In the effective field theory description one has a choice to either integrate out these massive states along with the whole tower of heavy string modes or leave them in the low energy effective action. The latter choice is appropriate when the masses are small and well below the cutoff of the effective theory. In this case we include these flat directions among the Q^I. The first choice is sensible whenever the masses are close to the string scale M_{str}. Once they are integrated out the gauge group is reduced and only gauge neutral states are left over. The important point is that flat directions that are charged change the low energy spectrum while gauge neutral flat directions are spectrum preserving. For a more detailed discussion of this distinction see for example [44].

[12] For further discussion about the Wilsonian Lagrangian see for example refs. [44,45].

[13] The important point here is that M_{Pl} is the field independent mass scale of the supergravity Lagrangian (14).

[14] More precisely, Z_r is the block of $Z_{\bar{I}J}$ corresponding to the matter multiplets in representation r.

We also abbreviated

$$
\begin{aligned}
T_a(r)\,\delta^{(a)(b)} &= \mathrm{Tr}_a T^{(a)} T^{(b)} \\
T(G_a) &= T_a(\text{adj. of } G_a) \\
b_a &= \sum_r n_r T_a(r) - 3\,T(G_a) \qquad (32) \\
c_a &= \sum_r n_r T_a(r) - T(G_a)\ ,
\end{aligned}
$$

where $T^{(a)}$ are the generators of the gauge group and n_r denotes the number of matter multiplets in representation r.

There are several points about eqs. (31) which need to be stressed:

- The effective gauge couplings are not harmonic functions of the moduli, that is $\partial_i \partial_{\bar{j}} g_a^{-2} \neq 0$. This failure of harmonicity is known as the holomorphic or Kähler anomaly. It implies that g_a^{-2} is not the real part of a holomorphic function; $g_a^{-2} = \mathrm{Re} f_a$ only holds at the tree level but not when higher loop corrections are included.[15]

- The non-harmonic differences $g_a^{-2} - \mathrm{Re} f_a$ only depend on the massless modes and their couplings. Therefore they can be computed entirely in the low energy effective field theory without any additional knowledge about the underlying fundamental theory.

- The Wilsonian couplings f_a are always holomorphic and only corrected at the one-loop level and non-perturbatively. These quantum corrections are induced by the heavy modes of the underlying fundamental theory.

- The axionic couplings still obey $\partial_{M^j} \theta_a = 8i\pi^2 \partial_{M^j} g_a^{-2}$ but $\partial_{M^j} \theta_a$ is no longer integrable whenever g_a^{-2} is non-holomorphic

---

[15]A similar situation has been found for the superpotential [48–50,44,51,52]. Beyond the tree level it is necessary to make a distinction between the effective Yukawa couplings and the holomorphic Wilsonian parameters of the superpotential. They coincide at the tree level but not beyond when massless modes are in the spectrum. The holomorphic superpotential W is by construction a Wilsonian quantity and obeys all the non-renormalization theorems of the previous section.

[53]. The θ-angles are not well defined when massless modes are present.

The effective gauge couplings $g_a(p^2)$ are physical quantities and hence must be invariant under all exact symmetries of the theory. This is certainly assured at the classical level but from eqs. (31) we learn that such an invariance is potentially spoiled at the quantum level. For example a holomorphic coordinate transformation

$$
\begin{aligned}
Q^I &\rightarrow \Sigma^I{}_J(M)\,Q^J \qquad (33) \\
Z_{\bar{J}I} &\rightarrow \bar{\Sigma}^{-1\,\bar{K}}_{\bar{J}}\,Z_{\bar{K}L}\Sigma_I^{-1\,L}
\end{aligned}
$$

or a Kähler transformation

$$
\begin{aligned}
\hat{K} &\rightarrow \hat{K} + F(M) + \bar{F}(\bar{M}) \\
W &\rightarrow W e^{-F(M)} \qquad (34)
\end{aligned}
$$

do not leave $g_a(p^2)$ of eqs. (31) invariant. This is a one-loop violation of two classical invariances and therefore they are often referred to as the σ-model and Kähler anomaly. However, these anomalies are harmonic functions of the moduli and therefore can be cancelled by assigning a new one-loop transformation law to the gauge kinetic functions f_a. Cancellation of the combined anomalies requires the Wilsonian couplings to transform as:

$$
f_a \rightarrow f_a - \frac{c_a}{8\pi^2}F - \sum_r \frac{T_a(r)}{4\pi^2}\ln\det\Sigma^{(r)}. \qquad (35)
$$

In that sense $f_a(M)$ can be viewed as local counterterms cancelling potential σ-model and Kähler anomalies [46,29–32,44,47].

Finally, based on ref. [54] one can write down an all-loop generalization of eqs. (31) [30,44]

$$
g_a^{-2}(p^2) = \mathrm{Re} f_a + \frac{b_a}{16\pi^2}\ln\frac{M_{\mathrm{Pl}}^2}{p^2} + \frac{c_a}{16\pi^2}\hat{K} \qquad (36)
$$

$$
+ \frac{T(G_a)}{8\pi^2}\ln g_a^{-2}(p) - \sum_r \frac{T_a(r)}{8\pi^2}\ln\det Z_r(p)\ ,
$$

where now $\hat{K}, Z_r(p)$ are the full loop-corrected couplings. Eqs. (36) are the solutions of the all-loop β-functions proposed in [54]

$$
\beta_a(g_a) \equiv p\,\frac{dg_a}{dp} = \frac{g_a^3}{16\pi^2}\frac{b_a + \sum_r T_a(r)\,\gamma_r}{1 - \frac{T(G_a)}{8\pi^2}g^2}\ , \qquad (37)
$$

where $\gamma_r := p\frac{d}{dp}\ln\det Z_r(p)$.

2.3.3. Gauge Couplings in String Theory

The next step is to apply the 'lessons' of the previous section to string theory and explicitly compute the low energy gauge couplings. First of all, the dilaton dependence of g_a can be determined by inserting eqs. (8),(25),(27) into (31) which results in

$$g_a^{-2} = k_a \mathrm{Re} S + \frac{b_a}{16\pi^2} \ln \frac{M_{\mathrm{str}}^2}{p^2} + \Delta_a(M, \bar{M}) \quad (38)$$

where

$$\Delta_a(M, \bar{M}) = \mathrm{Re} f_a^{(1)}(M) + \frac{c_a}{16\pi^2} \hat{K}^{(0)}(M, \bar{M})$$
$$- \sum_r \frac{T_a(r)}{8\pi^2} \ln \det Z_r^{(0)}(M, \bar{M}) , \quad (39)$$

and

$$M_{\mathrm{str}}^2 \sim \frac{M_{\mathrm{Pl}}^2}{S + \bar{S}} . \quad (40)$$

The precise numerical coefficient which relates the scales M_{str} and M_{Pl} is a matter of convention (see for example [55]) and the discussion here neglects all field independent corrections of g_a. From eqs. (38)–(40) we learn that the entire one-loop dilaton dependence of eq. (31) conspires into a $\ln(S + \bar{S})$ term with the one-loop beta function as a coefficient. This amounts to nothing but a universal change of the coupling's unification scale. In string theory the natural starting point for the renormalization of the effective gauge couplings is not M_{Pl} but rather the string scale M_{str}. That is, string theory and field theory naturally choose different cutoffs. Furthermore, the threshold corrections Δ_a are entirely independent of the dilaton ($\partial_S \Delta_a = 0$) (this fact can also be derived directly from $\mathcal{L}_{\mathrm{eff}}$ in the string frame eq. (20) [56]).

The moduli dependence of Δ_a can be computed in two different ways. First of all one can explicitly evaluate the one-loop string diagram (fig. 5) with two gauge boson vertex operators V_{v_m} as the external legs [55,53,57–60]. In order to do this computation one needs to know the exact massive spectrum which runs in the loop or in other words the CFT correlation functions have to be known.
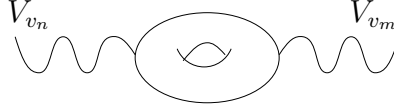

Figure 5. Two-point genus one string amplitude with two gauge bosons.

A second possibility to determine Δ_a makes use of the constraints implied by the holomorphy of f_a, its quantum symmetries and singularity structure. This second method will be discussed in some detail in the next section. It has the advantage that it can be used even if the CFT correlation functions are unknown. However, in some case the constraints on f_a are not strong enough to determine it uniquely but only up to a small number of undetermined numerical coefficients [59].

2.3.4. Orbifold compactifications

As an example we consider a specific class of orbifold compactification of the ten-dimensional heterotic string [61,53]. An orbifold is constructed from a smooth toroidal compactification by dividing the six-torus T^6 by a non-freely acting discrete group. In order to preserve $N = 1$ supersymmetry this discrete group should be a subgroup of $SU(3)$ and an isometry of T^6.

Here we focus on a particular subclass of such orbifold compactifications (factorizable orbifolds) where the geometrical moduli of (at least) one two-torus T^2 of T^6 are left unconstrained in the spectrum. An example of a factorizable orbifold is the $\mathbf{Z}_4$ orbifold with the generator $\theta = (i, i, -1)$ acting on the three complex coordinates of $T^6 = T^2 \times T^2 \times T^2$. The moduli of the third T^2 are

$$T = 2(\sqrt{g} - ib) ,$$
$$U = \frac{1}{g_{11}}(\sqrt{g} - ig_{12}) , \quad (41)$$

where g_{ij} is the background metric on T^2, $g = \det(g_{ij})$ and $b_{ij} = b\epsilon_{ij}$ is the antisymmetric tensor.[16]

---

[16]For a precise definition of factorizable orbifolds see ref. [47].

The tree level couplings of these toroidal moduli are given by [62–64,15,65,47]

$$\hat{K}^{(0)} = -\ln(T + \bar{T}) - \ln(U + \bar{U}) ,$$
$$Z_{I\bar{J}}^{(0)} = \delta_{I\bar{J}} (T + \bar{T})^{-q_I} (U + \bar{U})^{-q_I} , \qquad (42)$$

where the q_I are rational numbers depending on the twist sector of the orbifold; they can be found for example in ref. [47].

Factorizable orbifold compactifications always have an $SL(2, \mathbf{Z})_T \times SL(2, \mathbf{Z})_U$ quantum symmetry [66,8]. That is, the partition function as well as all correlation functions respect this symmetry to all orders in string perturbation theory. Such quantum symmetries commonly arise in string vacua and they are termed T-dualities. For the particular case at hand the $SL(2, \mathbf{Z})_T$ acts on the toroidal moduli according to:

$$T \rightarrow \frac{aT - ib}{icT + d} , \quad U \rightarrow U , \qquad (43)$$

where $a, b, c, d \in \mathbf{Z}$ and $ad - bc = 1$. The $SL(2, \mathbf{Z})_U$ has a similar action with T and U interchanged. (Further details of the modular group $SL(2, \mathbf{Z})$ are collected in appendix A.) Under the transformation (43) the Kähler potential of eq. (42) undergoes a Kähler shift of the form (34) with

$$F(T) = \ln(icT + d) . \qquad (44)$$

(The same shift is found for $SL(2, \mathbf{Z})_U$ but with T replaced by U.)[17] From eq. (42) one also infers that $Z_{I\bar{J}}^{(0)}$ transforms according to eq. (33) with

$$\Sigma^I{}_J = \delta^I{}_J (icT + d)^{-q_I} . \qquad (45)$$

Inserting (44) and (45) into (35) and using the fact that the dilaton S can be chosen modular invariant one finds that the one-loop corrections $f_a^{(1)}(T, U)$ have to transform like

$$f_a^{(1)}(T, U) \rightarrow f_a^{(1)}(T, U) - \frac{\alpha_a}{8\pi^2} \ln(icT + d) \qquad (46)$$

where

$$\alpha_a = \sum_I T_a(Q^I)(1 - 2q_I) - T_a(G) . \qquad (47)$$

---

[17] Note that although the tree level Kähler potential is corrected in string perturbation theory the holomorphic function $F(T)$ has to be exact to all orders of perturbation theory. This follows from the fact that the superpotential W is protected from any perturbative renormalization.

The logarithm of the Dedekind η-function (defined in appendix A) has precisely the transformation properties needed to satisfy (46). More specifically one has

$$\eta^2 \rightarrow \epsilon \eta^2 (icT + d) , \qquad (48)$$

where $\epsilon^{12} = 1$. Hence one infers

$$f_a^{(1)} = -\frac{\alpha_a}{8\pi^2} \ln[\eta^2(iT)\eta^2(iU)]$$
$$+ P_a[j(iT), j(iU)] , \qquad (49)$$

where P_a are modular invariant holomorphic functions of the moduli and thus can only depend on T and U through the modular invariant holomorphic j-function defined in appendix A.

In order to determine P_a we also need to know the singularities of f_a. Such singularities appear at points in the moduli space where some otherwise heavy states become massless. For factorizable orbifolds there are indeed additional massless states at $T = U$ (mod $SL(2, \mathbf{Z})$) but they are always neutral with respect to the low energy gauge group. Thus the gauge couplings do not develop a singularity at $T = U$ and consequently the f_a cannot have any singularities at finite T or U in the moduli space. On the other hand, in the limit $\mathrm{Re}\, T, \mathrm{Re}\, U \rightarrow \infty$ the theory decompactifies and a whole tower of Kaluza–Klein states turns massless. However, the corresponding singularity in the gauge couplings is constrained by the fact that there has to exist a region in the moduli space where the theory stays perturbative in the decompactification limit. This region is characterized by the requirement that both the four-dimensional gauge coupling g_4 and the six-dimensional gauge coupling g_6 are small. In the large radius limit the two coupling are related by

$$g_4^{-2} \sim r^2 g_6^{-2} , \qquad (50)$$

where $r = \sqrt{\mathrm{Re}\, T}$ (c.f. (41)) is the radius of the torus. Therefore, g_6 stays small for $g_4^2 \cdot \mathrm{Re}\, T$ fixed and small. g_4 stays small for $\mathrm{Re}\, S$ large and $\mathrm{Re}\, S \gg \Delta$ where this last condition merely states that the one-loop correction is small compared to the tree level value. This constrains Δ or equivalently $f_a(T, U)$ to grow at most linearly in T or

$$\lim_{\mathrm{Re}\, T \rightarrow \infty} f_a \rightarrow T . \qquad (51)$$

(The same argument also holds for U since the large U limit is $SL(2,\mathbf{Z})$ equivalent to the decompactification limit.) From its definitions in appendix A we learn that j and η are non-singular inside the fundamental domain and in the limit $T \to \infty$ their asymptotic behaviour is

$$\ln(\eta^2) \to -\frac{\pi}{12}T \ , \quad j \to e^{2\pi T} \ . \tag{52}$$

We see that the first term in eq. (49) does satisfy (51). The modular invariant $P_a[j(iT), j(iU)]$ also have to satisfy (51) but as we argued earlier they also have to be finite inside the modular domain. The unique solution to these three constraints is $P_a = \text{const.}$ [47].

To summarize, from the knowledge of the tree level couplings $\hat{K}^{(0)}$ and $Z^{(0)}$ and the fact that the modular symmetries $SL(2,\mathbf{Z})_T \times SL(2,\mathbf{Z})_U$ hold at any order in perturbation theory one uniquely determines

$$f_a(S,T,U) = k_a S - \frac{\alpha_a}{8\pi^2} \ln[\eta^2(iT)\eta^2(iU)] \tag{53}$$

up to (gauge group dependent) constants. Inserted into eq. (39) one finds

$$\Delta_a = -\frac{\alpha_a}{16\pi^2} \ln\left[|\eta(iT)|^4 \mathrm{Re}T \ |\eta(iU)|^4 \mathrm{Re}U\right]. \tag{54}$$

The exact same method presented here can also be applied to Calabi–Yau vacua with a low number of moduli and in many cases one is able to determine f_a (up to a universal gauge group independent factor) [67–71,47]. However, in some cases (for example for non-factorizable orbifolds) the constraints on f_a are not strong enough to determine it uniquely [59]. Instead one is left with a finite number of undetermined coefficients.

3. $N=2$ **String Vacua**

In this section we study the holomorphic couplings for heterotic and type II string vacua which have an extended $N=2$ space-time supersymmetry. In part this is motivated by the work of Seiberg and Witten who determined the exact non-perturbative low energy effective theory of an $N=2$ supersymmetric Yang–Mills theory [72]. In this case the entire low energy effective action is encoded in terms of a holomorphic prepotential $\mathcal{F}$. (The analysis of Seiberg and Witten

and its generalizations is nicely reviewed in the lectures of W. Lerche.) It was of immediate interest to also discover the Seiberg–Witten theory as the low energy limit of an appropriate string vacuum. (This aspect is reviewed in the lectures of S. Kachru.)

In this lecture we recall the structure of $N=2$ supergravity with special emphasis on the holomorphic prepotential and focus on perturbative properties of heterotic as well as type II $N=2$ string vacua.

3.1. $N=2$ **Supergravity**

$N=2$ extended space-time supersymmetry is generated by two (Weyl-) supercharges Q_α^A ($A=1,2$) which obey

$$\begin{aligned}\{Q_\alpha^A, \bar{Q}_{\dot\beta}^B\} &= 2\delta^{AB}\sigma^m_{\alpha\dot\beta}p_m \\ \{Q_\alpha^A, Q_\beta^B\} &= \epsilon_{\alpha\beta}\epsilon^{AB}\mathcal{Z} \ ,\end{aligned} \tag{55}$$

where $\mathcal{Z}$ is the central charge of the algebra.

The $N=2$ gravitational multiplet E contains the graviton, two gravitini $\psi_{m\alpha}^A$ and an Abelian vector boson γ_m called the graviphoton. In terms of $N=1$ multiplets it is the sum of the $N=1$ gravitational multiplet and a gravitino multiplet Ψ which contains a gravitino and an Abelian vector.[18] An $N=2$ vector multiplet contains a vector, two gaugini λ_α^A and a complex scalar ϕ; it consists of an $N=1$ vector multiplet V and a chiral multiplet Φ. Matter fields arise from $N=2$ hypermultiplets H which contain two Weyl fermions χ_α^A and four real scalars q^{AB}; they are the sum of two $N=1$ chiral multiplets. There are three further multiplets which all contain an antisymmetric tensor field and therefore also will accommodate the dilaton of string theory. First there is a vector-tensor multiplet VT which contains an Abelian vector, two Weyl fermions, the field strength of an antisymmetric tensor and a real scalar; it consists of an $N=1$ vector multiplet and a linear multiplet L [74–77]. The tensor multiplet T contains two Weyl fermions, the field strength of an antisymmetric tensor and three real scalars; it consists of an $N=1$ chiral multiplet and a linear multiplet [78]. Finally,

[18]This multiplet was constructed in ref. [73]; it cannot be consistently coupled in $N=1$ supersymmetry.

$N = 2$ multiplets	$N = 1$	spin
$E \sim (g_{mn}, \psi^A_{m\alpha}, \gamma_m)$	$\mathrm{E} \oplus \Psi$	$(2, \frac{3}{2}, 1)$
$V \sim (v_m, \lambda^A_\alpha, \phi)$	$V \oplus \Phi$	$(1, \frac{1}{2}, 0)$
$H \sim (\chi^A_\alpha, q^{AB})$	$\Phi \oplus \Phi$	$(\frac{1}{2}, 0)$
$VT \sim (v_m, H_m, \chi^A_\alpha, l)$	$V \oplus L$	$(1, \frac{1}{2}, 0)$
$T \sim (H_m, \chi^A_\alpha, \phi, l)$	$\Phi \oplus L$	$(\frac{1}{2}, 0)$
$\Pi \sim (H_m, H'_m, \chi^A_\alpha, l, l')$	$L \oplus L$	$(\frac{1}{2}, 0)$

Table 7
Multiplets of $N = 2$ supergravity in $d = 4$.

the double tensor multiplet Π contains two Weyl fermions, two field strengths of antisymmetric tensors and two real scalars; it consists of two linear multiplets.[19] All multiplets have four on-shell bosonic and fermionic degrees of freedom and we summarize their field content and spins in table 7.

As we discussed in the previous lectures an antisymmetric tensor is dual to a pseudoscalar in $d = 4$ and this translates into the following dualities among $N = 2$ multiplets

$$VT \sim V, \quad T \sim H, \quad \Pi \sim H . \qquad (56)$$

The dilaton which arises from the universal sector of the CFT is always accompanied by an antisymmetric tensor. More specifically, for any heterotic vacuum the dilaton is a member of a vector-tensor multiplet while the dilaton in type IIA (type IIB) vacua resides in a tensor (double-tensor) multiplet (table 8). This can be derived by dimensionally reducing the ten-dimensional string theories summarized in table 3.

As in lecture 2 we always choose to discuss the low energy effective theory in terms of the dual multiplets, that is we express the action solely in terms of the gravitational multiplet, the vector and hypermultiplets.

$N = 2$ supergravity severely constrains the interactions among these multiplets [79–85]. In particular, the complex scalars ϕ of the vector multiplets are coordinates on a special Kähler manifold $\mathcal{M}_V$ while the real scalars q^{AB} of the hypermulti-

string theory	dilaton multiplet
heterotic	$VT \sim V$
type IIA	$T \sim H$
type IIB	$\Pi \sim H$

Table 8
Dilaton multiplet.

plets are coordinates on a quaternionic manifold $\mathcal{M}_H$. Locally the two spaces form a direct product [81], i.e.

$$\mathcal{M} = \mathcal{M}_V \otimes \mathcal{M}_H . \qquad (57)$$

For their respective dimensions we abbreviate $\dim(\mathcal{M}_V) \equiv n_V$ and $\dim(\mathcal{M}_H) \equiv n_H$.

3.1.1. Special Kähler manifolds

A special Kähler manifold is a Kähler manifold whose geometry obeys an additional constraint [80].[20] One way to express this constraint is the statement that the Kähler potential K is not an arbitrary real function (as it was in $N = 1$ supergravity) but determined in terms of a holomorphic prepotential F according to[21]

$$K = -\ln \left(i \bar{Z}^\Sigma(\bar\phi) F_\Sigma(Z) - i Z^\Sigma(\phi) \bar{F}_\Sigma(\bar{Z}) \right) . \qquad (58)$$

The $Z^\Sigma, \Sigma = 0, \ldots, n_V$ are $(n_V + 1)$ holomorphic functions of the n_V complex scalar fields $\phi^I, I = 1, \ldots, n_V$ which reside in the vector multiplets. F_Σ abbreviates the derivative, i.e. $F_\Sigma \equiv \frac{\partial F(Z)}{\partial Z^\Sigma}$ and $F(Z)$ is a homogeneous function of Z^Σ of degree 2:

$$Z^\Sigma F_\Sigma = 2F . \qquad (59)$$

The Kähler metric G_{IJ} is obtained from eq. (15) with the Kähler potential (58) inserted.

The above description is again somewhat redundant. The holomorphic Z^Σ can be eliminated by an appropriate choice of coordinates and a choice of the Kähler gauge. The corresponding

---

[19] Dimensional reduction of type IIB string theory implies the existence of this multiplet (*c.f.* table 3) but as far as we know it has not been explicitly constructed yet. Therefore, we leave it as an exercise for the reader.

[20] For a review of special geometry, see for example ref. [86].
[21] Alternative definitions can be found in refs. [87,82,88] and their equivalence is discussed in ref. [89].

coordinates are called special coordinates and are defined by:

$$\phi^I = \frac{Z^I}{Z^0} \ . \qquad (60)$$

In these special coordinates the Kähler potential (58) reads

$$K = -\ln\Big(2(\mathcal{F} + \bar{\mathcal{F}}) - (\phi^I - \bar{\phi}^I)(\mathcal{F}_I - \bar{\mathcal{F}}_I)\Big), (61)$$

where $\mathcal{F}(\phi)$ is an arbitrary holomorphic function of ϕ^I related to $F(Z)$ via $F(Z) = -i(Z^0)^2 \mathcal{F}(\phi)$.

A lengthy but straightforward computation shows that using (58) the Riemann curvature tensor of special Kähler manifolds satisfies [90]

$$\begin{aligned} R_{I\bar{J}K\bar{L}} &= G_{I\bar{J}}G_{K\bar{L}} + G_{I\bar{L}}G_{K\bar{J}} \\ &\quad - e^{2K}W_{IKM}G^{M\bar{M}}\bar{W}_{\bar{M}\bar{J}\bar{L}} \ , \end{aligned} \qquad (62)$$

where $W_{IKM} \equiv \partial_I \partial_K \partial_M \mathcal{F}$.[22]

$N = 2$ supergravity not only constrains the Kähler manifold of the vector multiplets but also relates the gauge couplings to the holomorphic prepotential. More specifically, the gauge kinetic terms are

$$\mathcal{L} = -\frac{1}{4}g_{\Sigma\Lambda}^{-2}\,F_{mn}^{\Sigma}F^{mn\Lambda} - \frac{\theta_{\Sigma\Lambda}}{32\pi^2}\,F^{\Sigma}\tilde{F}^{\Lambda} + \dots, (63)$$

where F_{mn}^0 is the field strength of the graviphoton and

$$\begin{aligned} g_{\Sigma\Lambda}^{-2} &= \frac{i}{4}(\mathcal{N}_{\Sigma\Lambda} - \bar{\mathcal{N}}_{\Sigma\Lambda}) \ , \\ \theta_{\Sigma\Lambda} &= 2\pi^2(\mathcal{N}_{\Sigma\Lambda} + \bar{\mathcal{N}}_{\Sigma\Lambda}) \ . \end{aligned} \qquad (64)$$

$\mathcal{N}_{\Sigma\Lambda}$ is defined by:

$$\mathcal{N}_{\Sigma\Lambda} = \bar{F}_{\Sigma\Lambda} + 2i\,\frac{\mathrm{Im}F_{\Sigma\Omega}\,\mathrm{Im}F_{\Lambda\Xi}\,Z^{\Omega}Z^{\Xi}}{\mathrm{Im}F_{\Omega\Xi}\,Z^{\Omega}Z^{\Xi}} \ . \qquad (65)$$

Due to the second term of $\mathcal{N}_{\Sigma\Lambda}$ the gauge couplings are generically non-harmonic functions of the moduli. This is different from the situation in $N = 1$ since any Abelian factor in the gauge group can have a non-trivial mixing with

the graviphoton and this is origin of the non-harmonicity of the (Wilsonian) gauge couplings in $N = 2$ supergravity.

To summarize, both the Kähler potential and the Wilsonian gauge couplings of the vector multiplets in an $N = 2$ effective Lagrangian are determined by a single holomorphic function of the scalar fields ϕ – the prepotential $\mathcal{F}(\phi)$. The Wilsonian gauge couplings are non-harmonic already at the tree level.

It is again convenient to make a distinction between gauge neutral moduli scalars M^i which are members of Abelian vector multiplets and charged scalars Q^I which arise from non-Abelian vector multiplets.[23] That is, we split the scalars ϕ^I according to $\phi^I = (M^i, Q^I)$[24] and expand $\mathcal{F}$ and K as a truncated power series around $\langle Q^I \rangle = 0$ exactly as in $N = 1$

$$\mathcal{F} = h(M) \ + \ f_{IJ}(M)\,Q^I Q^J + \dots \ . \qquad (66)$$

Inserted into (61) one finds [75]

$$K = \hat{K}(M, \bar{M}) \ + \ Z_{IJ}(M, \bar{M})\,Q^I \bar{Q}^J + \dots \qquad (67)$$

where

$$\begin{aligned} \hat{K} &= -\ln\Big(2(h + \bar{h}) - (M^i - \bar{M}^i)(h_i - \bar{h}_i)\Big) \\ Z_{IJ} &= 4\,e^{\hat{K}}\mathrm{Re}f_{IJ}(M) \ . \end{aligned} \qquad (68)$$

The gauge couplings of any non-Abelian factor G_a also simplify since gauge invariance of eq. (63) requires $f_{IJ} = \delta_{IJ}\,f_a$ where here I and J label the vector multiplets of the factor G_a. Inserted into eqs. (64),(65) reveals that the (Wilsonian) gauge coupling of a non-Abelian factor is a harmonic function of the moduli

$$g_a^{-2} = \mathrm{Re}f_a(M) \ . \qquad (69)$$

As for $N = 1$ this only holds at the tree level. At the loop level the effective (non-Abelian) gauge

---

[22]The W_{IKM} are sometimes referred to as the Yukawa couplings since for a particular class of heterotic $N = 1$ vacua (vacua which have a global $(2, 2)$ worldsheet supersymmetry) they correspond to space-time Yukawa couplings [15].

[23]As we discussed in the previous section this distinction is somewhat ambiguous and involves a choice of the low energy degrees of freedom. In $N = 2$ the scalars in the Cartan subalgebra of a non-Abelian gauge factor G_a always are flat directions of the effective potential. Thus at a generic point in their moduli space G_a is broken to its maximal Abelian subgroup.

[24]As before we use the index I in two different ways and hope the reader will not be confused by this notation.

couplings again cease to be harmonic and instead obey the $N = 2$ analog of eq. (31) which is found to be [75]

$$g_a^{-2} = \mathrm{Re}f_a + \frac{b_a}{16\pi^2}\Big(\ln \frac{M_{Pl}^2}{p^2} + \hat{K}(M, \hat{M})\Big). \quad (70)$$

One way to derive this relation is to simply insert eqs. (68), (69) into eq. (31).[25]

The moduli of $N = 2$ supergravity can be scalars in either vector or hypermultiplets and the total moduli space is a locally a direct product as in eq. (57). However, from eq. (70) we learn that the effective gauge couplings only depend on the moduli M^i in vector multiplets but not on the moduli in hypermultiplets. This is a consequence of the fact that the $N = 2$ gauge fields couple to charged hypermultiplets in a minimal gauge covariant way but they do not have two derivative couplings with any of the neutral hypermultiplets [81] and hence the gauge couplings also cannot depend on the latter. However, the spectrum of charged hypermultiplets does enter into eqs. (70) in that they affect the β-function coefficients b_a.

3.2. $N = 2$ Heterotic Vacua

So far we reviewed the effective $N = 2$ supergravity without any reference to a particular string theory. The aim of this section is to determine the prepotential $\mathcal{F}$ for $N = 2$ heterotic vacua.[26]

3.2.1. $N = 2$ Non-Renormalization Theorems

As we already discussed, for heterotic vacua the dilaton is part of a vector-tensor multiplet but we choose to discuss it in terms of its dual vector multiplet. More precisely, the dilaton is the real part of the scalar component in the dual vector multiplet with the axion being the imaginary part. From the fact that the dilaton organizes the string perturbation theory together with product structure of the moduli space (57) one derives the following non-renormalization theorem [75,94]:

---

[25]As for $N = 1$ vacua this formula can also be viewed as an all-loop expression. In $N = 2$ the β-function is only corrected at one-loop in perturbation theory but not beyond.
[26]The presentation of this section closely follows ref. [75]; similar results were obtained in ref. [91]. (See also refs. [92, 93].)

(i) The moduli space of the hypermultiplets is determined at the string tree level and receives no further perturbative or non-perturbative corrections, i.e.

$$\mathcal{M}_H = \mathcal{M}_H^{(0)} \ . \quad (71)$$

A. Strominger [94] stressed the validity of this non-renormalization theorem beyond string perturbation theory. This is a consequence of the fact that this theorem only depends on unbroken $N = 2$ supersymmetry and the assignment of the dilaton multiplet. In particular it does not depend on the continuous PQ-symmetry which in part was responsible for the non-renormalization theorem of W and f_a in $N = 1$ heterotic vacua. However, it was precisely the breaking of the PQ-symmetry which allowed for a violation of the $N = 1$ non-renormalization theorem by non-perturbative effects.

On the other hand the PQ-symmetry (10) can be used to derive a second non-renormalization theorem in $N = 2$. The loop corrections of the prepotential $\mathcal{F}$ are organized in an appropriate power series expansion in the dilaton. But exactly as in the $N = 1$ case the holomorphy of $\mathcal{F}$ and the PQ-symmetry only allow a very limited number of terms. Repeating the analysis of section 2.2. one finds:

(ii) The prepotential $\mathcal{F}$ only receives contributions at the string tree level $\mathcal{F}^{(0)}$, at one-loop $\mathcal{F}^{(1)}$ and non-perturbatively $\mathcal{F}^{(NP)}$, i.e.

$$\begin{aligned} \mathcal{F} &= \mathcal{F}^{(0)}(S, M) + \mathcal{F}^{(1)}(M) \\ &\quad + \mathcal{F}^{(NP)}(e^{-8\pi^2 S}, M) \ . \end{aligned} \quad (72)$$

$\mathcal{F}^{(0)}$ is of order S, $\mathcal{F}^{(1)}$ is dilaton independent and $\mathcal{F}^{(NP)}$ is only constrained by the discrete PQ-symmetry (28). This second theorem is the 'cousin' of the $N = 1$ non-renormalization theorem for W and f_a but in $N = 2$ it is more powerful since $\mathcal{F}$ determines both the Wilsonian gauge coupling and the Kähler potential.

3.2.2. String Tree Level

The next step is to determine the tree level prepotential $\mathcal{F}^{(0)}$ using additional information that

we have at our disposal. All (perturbative) $N = 2$ heterotic vacua also satisfy

- The tree level gauge coupling of any non-Abelian factor G_a is the set by the VEV of the dilaton.[27] That is, $g^{-2} = \mathrm{Re}S$ or

$$f^{(0)} = S. \tag{73}$$

- The dilaton dependence of the tree level Kähler potential is constrained by the PQ-symmetry and the fact that the dilaton arises in the universal sector. Therefore it cannot mix with any other scalar field at the tree level and one necessarily has

$$K^{(0)} = -\ln(S + \bar{S}) + \tilde{K}(M, \bar{M}, Q, \bar{Q}). \tag{74}$$

Surprisingly, this separation of the dilaton piece together with the constraint (61) uniquely fixes $\mathcal{F}^{(0)}$ to be [95]:

$$\mathcal{F}^{(0)} = -S\left(\eta_{ij}M^i M^j - \delta_{IJ}Q^I Q^J\right), \tag{75}$$

where $\eta_{ij} = \mathrm{diag}(1, -1, \ldots, -1)$. Inserted into (61) the tree level Kähler potential is found to be

$$\begin{aligned} K^{(0)} &= -\ln(S + \bar{S}) \tag{76} \\ &- \ln\left(\eta_{ij}\mathrm{Re}M^i\mathrm{Re}M^j - \delta_{IJ}\mathrm{Re}Q^I\mathrm{Re}Q^J\right). \end{aligned}$$

The metric derived from this Kähler potential is the metric of the coset space

$$\mathcal{M}_V^{(0)} = \frac{SU(1,1)}{U(1)} \times \frac{SO(2, n_V - 1)}{SO(2) \times SO(n_V - 1)}, \tag{77}$$

where the first factor of the moduli space is spanned by the dilaton and the second factor by the other vector multiplets. Let us stress once more that this results generically holds for perturbative heterotic $N = 2$ vacua and is a consequence of the constraints implied by supergravity and the special properties of the dilaton couplings.

---

[27] This only holds for the perturbative gauge group. In the last lecture we will see that non-perturbative effects can enlarge the gauge group but with a different coupling to the dilaton [16].

3.2.3. Perturbative Corrections

The next step is to determine $\mathcal{F}^{(1)}$, i.e. the one-loop corrections of the prepotential. Inserting (66) into (72) we see that both h and f_{IJ} have their own loop expansion

$$\begin{aligned} h &= h^{(0)} + h^{(1)} + h^{(NP)}, \\ f_{IJ} &= f_{IJ}^{(0)} + f_{IJ}^{(1)} + f_{IJ}^{(NP)}, \tag{78} \end{aligned}$$

and thus one has to compute $f^{(1)}$ and $h^{(1)}$. However, for these two couplings one cannot derive a result as general as we just did for $\mathcal{F}^{(0)}$. Instead we again only consider a particular subclass of heterotic $N = 2$ vacua and apply the method developed in the previous section. In order to use some of the earlier results we focus on those vacua which have two moduli T and U and a perturbative quantum symmetry $SL(2, \mathbf{Z})_T \times SL(2, \mathbf{Z})_U$ acting on T and U as in eq. (113). To be slightly more specific, let us consider compactifications of the ten-dimensional heterotic string on $K3 \times T^2$ where T and U are the two toroidal moduli of T^2 and $K3$ is a four-dimensional Calabi–Yau manifold.[28]

T and U are the scalar components of two Abelian vector multiplets so that for this class of vacua the gauge group is

$$G = G' \times U(1)_T \times U(1)_U \times U(1)_S \times U(1)_\gamma. \tag{79}$$

G' is a gauge factor which we do not further specify since it depends on the particular vacuum under consideration.[29] However, the rank of G is again bounded by the central charge $\bar{c}_{int} = 22$. For $N = 2$ vacua there are two additional $U(1)$

---

[28] There are vacua which have an $SL(2, \mathbf{Z})_T \times SL(2, \mathbf{Z})_U$ quantum symmetry but which cannot be interpreted as geometrical compactifications. We do not get into these subtleties here.

[29] G' also varies over the moduli space. As we already remarked, in $N = 2$ the vector multiplets in the Cartan subalgebra of G' are flat directions and therefore away from the origin of moduli space G' is generically broken to some subgroup. For example, compactification on $K3 \times T^2$ in which the spin connection is embedded in the gauge connection of $K3$ in the standard way, has a gauge group $G = E_8 \times E_7 \times U(1)_T \times U(1)_U \times U(1)_S \times U(1)_\gamma$ at the origin of the vector multiplet moduli space which at a generic point is broken to $G = U(1)^{15} \times U(1)_T \times U(1)_U \times U(1)_S \times U(1)_\gamma$. For this family of vacua there are also 65 moduli in hypermultiplets.

gauge bosons in the universal sector corresponding to the graviphoton and the superpartner of the dilaton. Thus we have

$$\text{rank}(G) \ \leq \ 22 + 2 \ . \tag{80}$$

$h^{(1)}(T, U)$ and $f^{(1)}(T, U)$ can be determined by an appropriate string loop computation. Here we follow instead the same method developed for $N = 1$ vacua and use the quantum symmetries and singularity structure to determine $h^{(1)}(T, U)$ and $f^{(1)}(T, U)$. However, there is slight complication now since $h^{(1)}(T, U)$ does have singularities inside the fundamental domain. As we already stated earlier there are additional gauge neutral massless modes on the subspace $T = U$ and in $N = 2$ vacua they belong to Abelian vector multiplets. Such states induce a logarithmic singularity in the $U(1)$ gauge couplings and by supersymmetry also render the associated θ-angles ambiguous. From eqs. (63)–(66) we learn that h is directly related to the $U(1)$ gauge couplings while f_a encodes the gauge couplings of the non-Abelian factors G_a. As explained in the previous section the f_a are non-singular inside the fundamental domain.

In order to determine the transformation properties of $h^{(1)}(T, U)$ under the modular group one can use a general formalism developed in ref. [75]. Here we just state the result that $h^{(1)}$ has to be a modular form of weight -2 if it were nowhere singular. In the presence of singularities one has to allow for integer ambiguities of the θ-angles which results in

$$h^{(1)}(T, U) \rightarrow \frac{h^{(1)}(T, U) + \Xi(T, U)}{(icT + d)^2} \tag{81}$$

for an $SL(2, \mathbf{Z})_T$ transformation. (For $SL(2, \mathbf{Z})_U$ T and U are interchanged in eq. (81).) Ξ is an arbitrary quadratic polynomial in the variables $(1, iT, iU, TU)$ and parameterizes the most general allowed ambiguities in the θ-angles; Ξ obeys

$$\partial_T^3 \Xi = \partial_U^3 \Xi = 0 \ . \tag{82}$$

As we said, for $\Xi = 0$ $h^{(1)}$ is a modular form of weight $(-2, -2)$ with respect to $SL(2, \mathbf{Z})_T \times SL(2, \mathbf{Z})_U$ but for non-zero Ξ it has no good modular properties. However, from eqs. (82), (120) we learn that instead $\partial_T^3 h^{(1)}$ is a single valued modular form of weight $(4, -2)$ and similarly $\partial_U^3 h^{(1)}$ has weight $(-2, 4)$.

The singularities in the T, U moduli space arise along the critical line $T = U$ (mod $SL(2, \mathbf{Z})$) where two additional massless gauge fields appear and the $U(1)_T \times U(1)_U$ is enhanced to $SU(2) \times U(1)$. Further enhancement appears at $T = U = 1$, which is the intersection of the two critical lines $T = U$ and $T = 1/U$. In this case one has 4 extra gauge bosons and an enhanced gauge group $SU(2) \times SU(2)$. The intersection at the critical point $T = U = \rho = e^{2\pi i/12}$ gives rise to 6 massless gauge bosons corresponding to the gauge group $SU(3)$. Altogether we have:

$$\begin{array}{lll} T = U: & U(1)_T \times U(1)_U \rightarrow SU(2) \times U(1) \\ T = U = 1: & U(1)_T \times U(1)_U \rightarrow SU(2) \times SU(2) \\ T = U = \rho: & U(1)_T \times U(1)_U \rightarrow \ SU(3) \end{array}$$

The singularity of the prepotential near $T = U$ can be determined by purely field theoretic considerations [96]. For the case at hand one finds [96,72,75]

$$\begin{aligned} h^{(1)}(T \sim U) \ &= \ \frac{1}{16\pi^2} \, (T - U)^2 \ln(T - U)^2 \\ &\quad + \text{regular terms} \end{aligned} \tag{83}$$

where the coefficient $\frac{1}{16\pi^2}$ is set by the $SU(2)$ β-function. (The general derivation of eq. (83) is reviewed in detail in the lectures by W. Lerche; see also [75].) From eq. (83) we learn that $\partial_T^3 h^{(1)}$ and $\partial_U^3 h^{(1)}$ have a simple pole at $T = U$.

As we already discussed in section 2, in the decompactification limit $(T, U \rightarrow \infty)$ the gauge coupling cannot grow faster than a single power of T or U and the exact same argument also applies for $h^{(1)}$. Since the moduli dependence of the gauge coupling is related to the second derivative of h we learn that $\partial_T \partial_U h^{(1)}$, $\partial_T^2 h^{(1)}$ and $\partial_U^2 h^{(1)}$ cannot grow faster than T or U in the decompactification limit. Hence

$$\partial_T^3 h^{(1)} \rightarrow \text{const.} \ , \quad \partial_U^3 h^{(1)} \rightarrow \text{const.} \tag{84}$$

for either $T \rightarrow \infty$ or $U \rightarrow \infty$. The properties of $h^{(1)}$ which we have assembled so far can be combined in the Ansatz

$$\partial_T^3 h^{(1)} = \frac{X_{-2}(U) \, Y_4(T)}{j(iT) - j(iU)} \ , \tag{85}$$

where $X_{-2}(U)$ and $Y_4(T)$ are modular forms of weight $(0, -2)$ and $(4, 0)$, respectively. $X_{-2}(U)$ and $Y_4(T)$ cannot have any pole inside the modular domain while for large T, U they have to obey

$$U \to \infty : \quad \frac{X_{-2}(U)}{j(iU)} \quad \to \text{const.} ,$$
$$T \to \infty : \quad \frac{Y_4(T)}{j(iT)} \quad \to \text{const.} . \quad (86)$$

From the theory of modular forms (see appendix A) one infers that these properties uniquely determine $X_{-2} = E_4 E_6 / \eta^{24}$ and $Y_4 = E_4$; inserted into (85) yields

$$\partial_T^3 h^{(1)} = \frac{1}{2\pi} \frac{E_4(iT) \, E_4(iU) \, E_6(iU)}{[j(iT) - j(iU)] \, \eta^{24}(iU)} , \quad (87)$$

where the coefficient is determined by eq. (83) or rather the $SU(2)$ β-function. The same analysis repeated for $\partial_U^3 h^{(1)}$ reveals

$$\partial_U^3 h^{(1)} = \frac{-1}{2\pi} \frac{E_4(iU) \, E_4(iT) \, E_6(iT)}{[j(iT) - j(iU)] \, \eta^{24}(iT)} . \quad (88)$$

The analysis just performed only determines the third derivatives of $h^{(1)}$ because these are modular forms. $h^{(1)}$ itself has been calculated in ref. [97] by explicitly calculating the appropriate string loop diagram.[30] An intriguing relation with hyperbolic Kač–Moody algebras and Borcherds denominator formula was found. Unfortunately, a review of these exciting developments is beyond the scope of these lectures.

In addition to the transformation law of h (eq. (81)) the general formalism of ref. [75] also reveals that the $N = 2$ dilaton is no longer invariant at the quantum level. Instead, under an $SL(2, \mathbf{Z})_T$ transformation one finds

$$S \to S + \frac{1}{2} \partial_T \partial_U \Xi - \frac{ic\partial_U(h^{(1)} + \Xi)}{2(icT + d)} + \text{const.} \quad (89)$$

This somewhat surprising result can be understood from the fact that in perturbative string theory the relation between the dilaton and the vector-tensor multiplet is fixed. However, the duality relation between the vector-tensor multiplet and its dual vector multiplet containing S is not fixed but suffers from perturbative corrections in

both field theory and string theory. Nevertheless, it is possible to define an invariant dilaton by

$$S^{\text{inv}} \equiv S - \frac{1}{2} \partial_T \partial_U h^{(1)} - \frac{1}{8\pi^2} \ln[j(iT) - j(iU)] . \quad (90)$$

The last term is added such that S^{inv} is finite so that altogether S^{inv} is modular invariant and finite. However, S^{inv} it is no longer an $N = 2$ special coordinate.[31]

We are now ready to determine the one-loop correction $f^{(1)}$. As before we demand that the effective gauge couplings (70) remain invariant under $SL(2, \mathbf{Z})_T \times SL(2, \mathbf{Z})_U$. Using (70),(44),(34) one finds that this requires the transformation property

$$f_a(S, T, U) \to f_a(S, T, U) - \frac{b_a}{8\pi^2} \ln(icT + d) , \quad (91)$$

where f_a is the entire function and not only the one-loop contribution. Using eqs. (89)-(91) together with the tree level contribution (73) one finds (again up to a constant)

$$f_a(S, T, U) = S^{\text{inv}} - \frac{b_a}{8\pi^2} \ln[\eta^2(iT)\eta^2(iU)] \quad (92)$$

or equivalently

$$f_a^{(1)} = \frac{-b_a}{8\pi^2} \ln[\eta^2(iT)\eta^2(iU)] + (S^{\text{inv}} - S). \quad (93)$$

To summarize, as for the $N = 1$ factorizable orbifolds we managed to use the perturbative quantum symmetries and the singularity structure to determine the one-loop correction of the gauge couplings. For the $U(1)$ factors we did not completely determine $h^{(1)}$ but only its third derivatives. $h^{(1)}$ itself can be computed using the formalism developed in ref. [97]. Further generalization to different classes of $N = 2$ string vacua can be found for example in refs. [91,104,105,97,98,101,102].

Up to now we mostly concentrated on the supersymmetric gauge couplings because of their special analytic properties. In addition, there is a class of higher derivative curvature terms whose couplings g_n are also determined by holomorphic

---

[30]See also refs. [98–103].

[31] In $N = 1$ it is always possible to keep the dilaton superfield chiral and modular invariant by an appropriate holomorphic field redefinition.

functions $\mathcal{F}_n(S, M^i)$ of the dilaton and the moduli [67,106,107].[32] (The properties of these couplings are the subject of the lectures by K.S. Narain.) More specifically, terms of the type

$$\mathcal{L} \sim g_n^{-2} R^2 G^{2n-2} + \dots , \qquad (94)$$

where R is the Riemann tensor and G the field strength of the graviphoton are governed by couplings g_n that are almost harmonic

$$g_n^{-2} = \mathrm{Re}\mathcal{F}_n(S, M^i) + \mathcal{A}_n . \qquad (95)$$

At the tree level $A_n = 0$ holds and thus g_n^{-2} is a harmonic function. However exactly as for the gauge coupling the g_n^{-2} cease to be harmonic as soon as quantum corrections are included, or in words a holomorphic anomaly $A_n \neq 0$ is induced [67]. The particular form of the anomaly is not of immediate concern here but can be found for example in refs. [67,27,108] and K.S. Narain's lectures. The same argument we gave earlier for f_a and the prepotential $\mathcal{F}$ implies that also the couplings $\mathcal{F}_n(S, M^i)$ can be expanded in powers of the dilaton and they have to respect the PQ-symmetry. Thus, one has analogously

$$\begin{aligned} \mathcal{F}_n &= \mathcal{F}_n^{(0)}(S, M^i) + \mathcal{F}_n^{(1)}(M^i) \\ &\quad + \mathcal{F}_n^{(NP)}(e^{-8\pi^2 S}, M^i) , \end{aligned} \qquad (96)$$

where $\mathcal{F}_n^{(0)}$ is the tree level term, $\mathcal{F}_n^{(1)}$ the one-loop correction and $\mathcal{F}_n^{(NP)}$ the non-perturbative contributions. Furthermore, the continuous PQ-symmetry (13) only allows a dilaton dependence for $n = 1$ and one finds

$$\mathcal{F}_1^{(0)} = 24\, S, \quad \mathcal{F}_{n>1}^{(0)} = \text{const.} , \qquad (97)$$

where for $\mathcal{F}_1^{(0)}$ a convenient normalization has been chosen. For specific string vacua, some of the $\mathcal{F}_n$ have been computed in refs. [27,108–111,101].

3.3. Left-Right Symmetric Type II Vacua

So far we focussed on heterotic $N = 2$ vacua and in particular on the moduli space of their vector multiplets. Now we shift our attention to type II vacua but with the additional input that there is a symmetry between the left and right moving CFT; this assumption considerably simplifies the analysis.[33] In $d = 4$ such vacua have a universal sector with $c_{\mathrm{st}} = \bar{c}_{\mathrm{st}} = 6$ that contains the gravitational multiplet and the dilaton multiplet which we already discussed in section 3.1. and table 8. The internal sector has $c_{\mathrm{int}} = \bar{c}_{\mathrm{int}} = 9$ and if it is left-right symmetric it also has $(2, 2)$ global worldsheet supersymmetry. This corresponds to a compactification of the ten-dimensional type II string on a Calabi–Yau threefold. (Calabi–Yau manifolds are reviewed in the lectures by R. Plesser; see also appendix B and for example refs. [1,114].)

The massless spectrum of a type II vacuum compactified on a Calabi–Yau threefold Y is characterized by the Hodge numbers $h_{1,1}$ and $h_{1,2}$ (appendix B). For type IIA one finds [115,116] $h_{1,1} + h_{1,2}$ complex massless scalar fields in the $NS - NS$ sector and $h_{1,1}$ Abelian vectors together with $h_{1,2}$ complex scalars in the $R - R$ sector. These states (together with their fermionic partners) combine into $h_{1,1}$ vector multiplets and $h_{1,2}$ hypermultiplets. The total number of multiplets therefore is $n_V = h_{1,1}$, $n_H = h_{2,1} + 1$ where the extra hypermultiplet counts the dilaton multiplet of the universal sector. For type IIB vacua one also has $h_{1,1} + h_{1,2}$ complex massless scalar fields in the $NS - NS$ sector but now $h_{1,2}$ Abelian vectors together with $h_{1,1}$ complex scalars in the $R - R$ sector [115,116]. Hence, $n_V = h_{2,1}$ and $n_H = h_{1,1} + 1$ holds for the type IIB theory. The gauge group is always Abelian and given by $(h_{1,1} + 1)\, U(1)$ factors in type IIA and $(h_{1,2} + 1)\, U(1)$ factors in type IIB (the extra $U(1)$ is the graviphoton in the universal sector). We summarize the spectrum of type II vacua in table 9.

As we see the role of $h_{1,1}$ and $h_{1,2}$ is exactly interchanged between type IIA and type IIB. Therefore, compactification of type IIA on a Calabi–Yau threefold Y is equivalent to compactification of type IIB on the mirror Calabi–Yau $\tilde{Y}$ (see appendix B). This is an example of a perturbative equivalence of two entire classes of string vacua. The recent developments about string du-

---

[32]The prepotential $\mathcal{F}$ as well as these higher derivative couplings $\mathcal{F}_n$ arise from chiral integrals in $N = 2$ superspace.

[33]More general type II vacua are discussed for example in refs. [112,113].

	IIA	IIB
n_H	$h_{1,2}+1$	$h_{1,1}+1$
n_V	$h_{1,1}$	$h_{1,2}$
G	$U(1)^{h_{1,1}+1}$	$U(1)^{h_{1,2}+1}$

Table 9
Massless spectrum in type II vacua.

alities show that many more of such equivalences do exist but they often only hold when the non-perturbative corrections are also taken into account. (See lectures by S. Kachru, R. Plesser and J. Schwarz.) We briefly return to this aspect in the next lecture.

We already discussed the geometry of the moduli space of $N = 2$ supergravity in section 3.1. For type II vacua there is a special feature at the string tree level. The $h_{1,1}$ and $h_{1,2}$ complex scalars by themselves each form a special Kähler manifold. This is an immediate consequence of table 9 in that they can both can be members of vector multiplets and thus their geometry must be special Kähler. However, when they sit in hypermultiplets they pair up with additional scalars from the $R-R$ sector and this renders the combined geometry quaternionic. Thus it cannot be an arbitrary quaternionic geometry but must originate from a special Kähler geometry. This fact has been termed the c-map in ref. [116] and it implies that also this 'special' quaternionic geometry is characterized by a holomorphic prepotential.[34]

However, the special quaternionic geometry only appears at the string tree level. This is due to the fact that in type II vacua the dilaton resides in a hypermultiplet and thus there is an analogous 'type II' version of the non-renormalization theorem (i) of section 3.2.1. It states that the moduli space of the vector multiplets is exact and not corrected either perturbatively or non-perturbatively while $\mathcal{M}_H$ does receive quantum corrections. In other words, for type II vacua one

has[35]

$$\mathcal{M}_V = \mathcal{M}_V^{(0)} \ . \tag{98}$$

The quaternionic geometry of both type II and heterotic vacua has not been studied in much detail[36] and most investigations focussed on the special Kähler geometry of the vector multiplets. For Calabi–Yau threefolds Y the prepotential $\mathcal{F}$ enjoys some additional properties. Of particular interest is the expansion around geometrically large Calabi-Yau manifolds which is also referred to as the large radius or rather large volume limit. In this limit the position in $\mathcal{M}_V$ is given by the complexified Kähler form $B+iJ$ which can be decomposed into a basis $e_\alpha \in H^{1,1}(Y, \mathbf{Z})$ according to

$$B + iJ = \sum_{\alpha=1}^{h_{1,1}} (B+iJ)_\alpha e_\alpha = t_\alpha e_\alpha \ , \tag{99}$$

where B_α are the moduli corresponding to the antisymmetric tensor and J_α the moduli which are associated to the deformation of the metric; t_α are the corresponding complex parameters ($\alpha = 1, \ldots, h_{1,1}$) which are also $N = 2$ special coordinates. In the large radius limit the moduli B_α are periodic variables and thus enjoy a discrete 'PQ-like' symmetry. More precisely, one finds that the low energy effective theory is invariant under $B_\alpha \to B_\alpha + 1$ or equivalently

$$t_\alpha \to t_\alpha + 1 \ . \tag{100}$$

In the large volume limit the prepotential obeys

$$\mathcal{F} = -\frac{i}{6} d_{\alpha\beta\gamma} t^\alpha t^\beta t^\gamma$$
$$+ \text{ worldsheet instantons } , \tag{101}$$

where $d_{\alpha\beta\gamma}$ are the classical intersection numbers of the $(1,1)$ forms defined by

$$d_{\alpha\beta\gamma} := \int_Y e_\alpha \wedge e_\beta \wedge e_\gamma \ . \tag{102}$$

Using mirror symmetry [122–124] the entire prepotential $\mathcal{F}$ can be computed for string vacua with small number of vector multiplets (see for

---

[34]The special quaternionic geometry is also called 'dual quaternionic'.

[35] As in the heterotic case, this non-renormalization theorem also holds non-perturbatively.
[36]See, however, refs. [115–121,83–85].

example [125–130,68–71]). In the limit $t_\alpha \to \infty$ the contributions from worldsheet instantons vanish.

The higher derivative couplings $\mathcal{F}_n$ which were defined in eqs. (94),(95) and which similarly only depend on the vector multiplets, also obey the type II non-renormalization theorem. For these couplings one finds the additional curiosity that they are proportional to the genus n topological partition functions of a twisted Calabi–Yau σ–model [67]. That is, they receive a contribution only at a fixed string loop order and satisfy certain recursion relations expressing the holomorphic anomaly of a genus n partition function to the partition function of lower genus [67]. The properties of these couplings both in type II and the heterotic string are reviewed in detail in K.S. Narain's lectures. In the large radius limit they obey

$$
\begin{aligned}
\mathcal{F}_1 &= -\frac{i}{4\pi} \sum_\alpha t^\alpha \int_Y e_\alpha \wedge c_2(Y) \\
&\quad + \text{worldsheet instantons} \qquad (103) \\
\mathcal{F}_{n>1} &= \text{const.} + \text{worldsheet instantons}
\end{aligned}
$$

where c_2 is the second Chern class of the Calabi–Yau manifold.

4. Heterotic-Type II Duality

So far we exclusively focussed on perturbative properties of string vacua although we noticed a number of non-renormalization theorems which also hold non-perturbatively. During the past two years it has become clear that there are much more intricate relations between (classes of) different string vacua than had hitherto been imagined. In particular, it is believed that string vacua which look rather different in string perturbation theory can yet be equivalent when all (perturbative and non-perturbative) quantum corrections are taken into account. By now there is whole web of relations among perturbatively distinct string vacua. These relations strongly depend on the amount of space-time supersymmetry and the number of space-time dimensions. This web of interrelations has been discussed in detail in the lectures of J. Schwarz. Here we focus on one particular correspondence namely the heterotic –

type II duality in $d = 4$ with $N = 2$ supersymmetry. This duality is also the subject of the lectures of S. Kachru and in this closing section we only briefly outline how the perturbative properties found earlier have been used to support the conjecture of this particular duality.

The conjecture states that $N = 2$ heterotic vacua are quantum equivalent to $N = 2$ type IIA vacua and vice versa [26,131].[37] On face value this conjecture seems unlikely to hold. First of all, for heterotic vacua the rank of the gauge group is bounded by the central charge to be less than 24 (eq. (80)) while in type II vacua the rank can certainly be much larger since both Hodge numbers easily exceed 22 (*c.f.* table 9). Furthermore, we saw that the heterotic vacua have large non-Abelian gauge groups at special points in their moduli space while type II A vacua only have an Abelian gauge group. However, the analysis of Seiberg and Witten [72] taught us that asymptotically free non-Abelian gauge groups generically do not survive non-perturbatively but instead are broken to their Abelian subgroups.[38]

It is not completely clear yet that the heterotic–type II duality holds for the whole space of heterotic and type II vacua or only on a (well defined) subspace. So far specific examples of string vacua or classes of string vacua have been proposed to be non-perturbatively equivalent. For such vacua – which are also called a 'dual pair' – the low energy effective theories have to be identical when all quantum corrections are taken into account. In particular their moduli spaces have to coincide, i.e.

$$
\begin{aligned}
\mathcal{M}_V^{het} &= \mathcal{M}_V^{IIA} \,, \\
\mathcal{M}_H^{het} &= \mathcal{M}_H^{IIA} \,. \qquad (104)
\end{aligned}
$$

Since the prepotential depends on the vector multiplets in both theories the equality of the moduli

---

[37] Since there is already the perturbative relation that type IIA vacua compactified on a Calabi–Yau manifold Y are equivalent to type IIB vacua compactified on the mirror $\tilde{Y}$ one really has a triality heterotic $\sim$ type IIA $\sim$ type IIB where the first equivalence only holds non-perturbatively while the second is a perturbative equivalence.

[38] Conversely one has to show that in a particular limit corresponding to the heterotic weak coupling limit a type II vacuum can have a non-Abelian enhancement of its gauge group [132,133].

spaces amounts to the equality of the prepotential

$$\mathcal{F}_{het} = \mathcal{F}_{IIA} . \tag{105}$$

This should not only be true for the prepotential but also hold for the higher derivative couplings $\mathcal{F}_n$. (See K.S. Narain's lectures.)

The proof of this conjecture is problematic since the prepotential on the heterotic side $\mathcal{F}_{het}$ is only known perturbatively, that is in a weak coupling expansion. For a heterotic vacuum weak coupling corresponds to large S and hence there has to be a type II modulus in a vector multiplet which is identified with the heterotic dilaton. It is immediately clear that this type II modulus cannot be the type II dilaton which always comes in a hypermultiplet. Instead it has to be one of the $h_{1,1}$ Kähler deformations of the Calabi–Yau threefold in the large radius limit.[39] Thus one is interested in identifying this dual type II partner t^s of the heterotic dilaton S. From the discrete PQ-symmetries (28),(100) one immediately infers that the relation must be

$$t^s \equiv 4\pi i S . \tag{106}$$

Once t^s has been identified one can expand the type IIA prepotential $\mathcal{F}_{IIA}$ in a t^s perturbation expansion around large t^s, i.e.

$$\mathcal{F}_{II} = \mathcal{F}_{II}(t^s, t^i) + \mathcal{F}_{II}(t^i) + \mathcal{F}_{II}(e^{-2\pi i t^s}, t^i) \tag{107}$$

where we use the notation $t^\alpha = (t^s, t^i)$. This expansion can be compared to the perturbative expansion of the heterotic prepotential. In particular one has to find

$$\mathcal{F}_{II}(t^s, t^i) + \mathcal{F}_{II}(t^i) = \mathcal{F}_{het}^{(0)}(S, \phi^I) + \mathcal{F}_{het}^{(1)}(\phi^I) \tag{108}$$

(up to an overall normalization which is convention and has to be adjusted appropriately). Let us reiterate that the left hand side of this equation is determined at the tree level whereas the right hand side sums perturbative contributions at the tree level and at one-loop.

Eq. (108) has been verified for number of explicit string vacua. Typically, these are vacua

---

[39]This immediately tells us that the appearance of the vector-tensor multiplet must be an artifact of heterotic perturbation theory. Similarly, the type II tensor multiplet is an artifact of type II perturbation theory.

with a small number of vector multiplets or low $h_{1,1}$ where the perturbative prepotential is known on both sides. These examples can be found in S. Kachru's lectures or in refs. [27,28,134–139,98,102].

Eq. (108) checks the spectrum of the two vacua and their quantum symmetries. However, it turns out that this does not uniquely identify a dual pair. Instead, there can be an entire class of vacua where each member satisfies eq. (108) but nevertheless their non-perturbative prepotentials $\mathcal{F}^{(NP)}$ are different [140,141,138,139]. For these cases additional, non-perturbative information is necessary to uniquely identify a dual pair. Unfortunately we cannot go into any further detail about such examples.

It has also been shown that on the heterotic side one discovers the Seiberg–Witten theory in the field theory limit $M_{str} \to \infty$ [26,142–144,105]. (This is reviewed in the lectures by W. Lerche.) Finally, the matching of the higher derivative gravitational couplings can be found in refs. [27,108–111] and K.S. Narain's lectures.

Apart from the specific checks we just mentioned it is of interest to determine some more generic properties of the heterotic–type II duality. That is, one would like to study in general the relation between a dual pair (or a class of dual pairs) as well as the space of string vacua for which this conjectured duality holds. From all our previous discussion it is clear that generic properties should involve the heterotic dilaton since it couples universally for all heterotic vacua. Indeed, from eqs. (75),(101),(108) one infers that the Calabi–Yau intersection numbers of a dual type IIA vacuum have to obey

$$d_{sss} = 0 , \qquad d_{ssi} = 0 \quad \forall i , \tag{109}$$

and

$$\text{sign}(d_{sij}) = (+, -, \ldots, -) = \text{sign}(\eta_{ij}) . \tag{110}$$

In addition, eqs. (97), (103) imply

$$\int e_S \wedge c_2(Y) = 24 . \tag{111}$$

These conditions are not unknown in the mathematical literature. They are the statement that

the Calabi–Yau manifold has to be a $K3$-fibration [28,145]. That is, the Calabi–Yau manifold is fibred over a $\mathbf{P^1}$ base with fibres that are $K3$ manifolds. The size of the $\mathbf{P^1}$ is parameterized by the modulus t^s which is the type II dual of the heterotic dilaton.[40] Over a finite number of points on the base, the fibre can degenerate to something other than $K3$ and such fibres are called singular. The other Kähler moduli t^i are either moduli of the $K3$ fibre or of the singular fibres. In general one finds

$$\mathrm{sign}(d_{sij}) = (+, -, \ldots, -, 0, \ldots, 0) \ , \qquad (112)$$

where the non-vanishing entries correspond to moduli from generic $K3$ fibres while the zeros arise from singular fibres. Since a $K3$ has at most 20 moduli the non-vanishing entries have to be less than 20.[41] Comparing eqs. (110) and (112) one concludes that type II Calabi-Yau compactification in the large radius limit can be the dual of perturbative heterotic vacua if they are $K3$-fibration with all moduli corresponding to generic fibres. This class of type II vacua is automatically consistent with the heterotic bound on the rank of the gauge group (80).

Of course, one immediately asks the question what can be the role of all the other type II vacua. From eqs. (75) and (112) we learn that the $(1,1)$ moduli of singular fibres have no counterpart in perturbative heterotic vacua. If there were heterotic moduli with such couplings they would not couple properly to the (heterotic) dilaton and furthermore violate the bound (80). However, Witten observed that heterotic string vacua in six space-time dimensions have singularities when gauge instantons shrink to zero size [16]. He argued that these singularities are caused by a non-perturbative enhancement of the gauge group which opens up at the point of the zero-size instanton. He also showed that these non-perturbative gauge fields do not share the canonical coupling to the dilaton. In fact, upon compactification to $d = 4$ one can show that the

scalars of these non-perturbative vector multiplets couple precisely like type II moduli corresponding to singular fibres [141,139].

This leads to the following general picture of the heterotic–type II duality. The space of left-
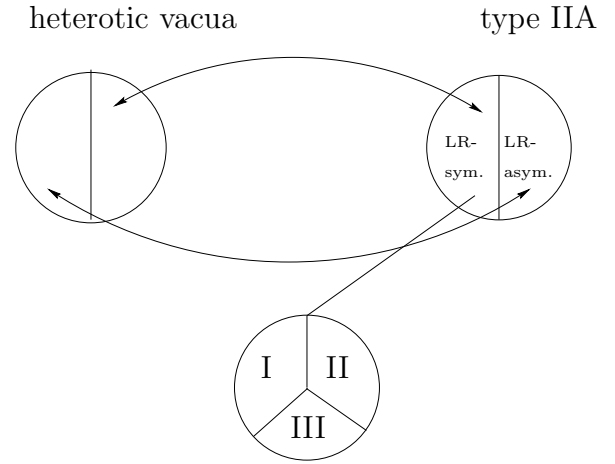

Figure 6. Heterotic – type II duality.

right symmetric vacua can be partioned into three different regions (fig. 6).

I $K3$–fibration with moduli only from generic fibres. All such vacua should be non-perturbatively equivalent to perturbative heterotic vacua.

II $K3$–fibration with moduli also from singular fibres. For such vacua a dual type II candidate for the heterotic dilaton does exist but some of the other moduli do not couple perturbatively; they have to arise from vector multiplets which cannot be seen in heterotic perturbation theory.

III Calabi–Yau manifolds which are not $K3$–fibration. Such vacua have no dual candidate for the heterotic dilaton and thus they cannot be the dual of a weakly coupled heterotic vacuum. However, they could well

---

[40] For a more detailed discussion about $K3$-fibrations see for example [28,113,145,146].

[41] There is a possible subtlety here since this counts only geometrical $K3$ moduli. However, it is conceivable that quantum effects raise this number up to 22 [147].

be the dual of heterotic vacua which have no weak coupling limit with a dilaton that is frozen at strong coupling. In that sense this might be the most interesting class of type II vacua.

From refs. [94,148] we learned that transitions among string vacua can occur once non-perturbative states are taken into the effective action near singularities of the moduli space. In the same spirit it is particularly interesting to study transitions between vacua that cross any of the boundaries in fig. 6. Examples of such transitions have already been observed in refs. [149,150] and are reviewed in the lectures of R. Plesser.

A. The modular group $SL(2, \mathbf{Z})$

The modular group $SL(2, \mathbf{Z})$ [8,151–153] enters string theory in various places. First of all, it is the group of reparametrizations of a genus one worldsheet which are not continuously connected to the identity. In the definition of the physical one-loop scattering amplitudes this additional gauge freedom has to be taken care of. Topologically a one-loop string amplitude is a torus and the modular group acts on the complex structure of this torus. However, tori in space-time also enter in string compactifications as for example in the construction of orbifold vacua. As a consequence the modular group also appears as a (quantum) symmetry of specific space-time effective theories.

The modular group is defined by the following transformation on the complex modulus T

$$T \rightarrow \frac{aT - ib}{icT + d} , \qquad ad - bc = 1 , \qquad (113)$$

where $a, b, c, d \in \mathbf{Z}$.[42] It has two generators $\mathcal{S}, \mathcal{T}$ which act as

$$\begin{aligned} \mathcal{T} : T &\rightarrow T + i \\ \mathcal{S} : T &\rightarrow -1/T \end{aligned} \qquad (114)$$

The transformation (113) maps the $\text{Re}T > 0$ region in a rather complicated way onto itself. However, one can define a fundamental region by the requirement that every point of the $\text{Re}T > 0$ complex plane is mapped into this region in a unique way. One conventionally chooses:

$$\Gamma = \{-\frac{1}{2} \leq \text{Im}T \leq \frac{1}{2}, \text{Re}T > 0, |T|^2 > 1\} \quad (115)$$

as the fundamental domain (figure 7). No two distinct points in Γ are equal under a modular transformation. There are two fixed points of the map (113) on this fundamental domain, namely $T = 1$ and $T = \rho \equiv e^{\frac{i\pi}{6}}$.
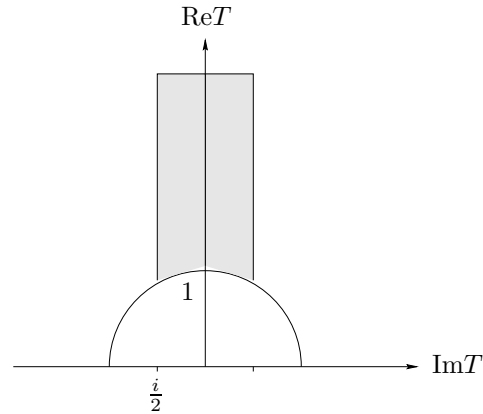

Figure 7. Fundamental region Γ of the modular group.

A modular form $F_r(T)$ of weight r is defined to be holomorphic and to obey the transformation law

$$F_r(T) \rightarrow (icT + d)^r F_r(T) . \qquad (116)$$

---

[42] It is common to choose a different convention for T where real and imaginary part are exchanged. More precisely, for $\tau = iT$ one has $\tau \rightarrow \frac{a\tau + b}{c\tau + d}$.

One can show that there are no modular forms of weight 0 and 2 while at weight 4 and 6 one has the Eisenstein functions

$$
\begin{aligned}
E_4(q) &\equiv 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n} \\
&= 1 + 240q + 2160q^2 \dots , \quad (117) \\
E_6(q) &\equiv 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1 - q^n} \\
&= 1 - 504q - 16632q^2 \dots ,
\end{aligned}
$$

where $q \equiv e^{-2\pi T}$. Both function have no pole (including $T = \infty$) on the entire fundamental domain; E_4 has exactly one simple zero at $T = \rho$ while E_6 has one simple zero at $T = 1$. One can construct modular forms of arbitrary even weight from products of these two Eisenstein functions.

A modular form which vanishes at $T = \infty$ is called a cusp form. There is no cusp form of weight $r < 12$ and for $r = 12$ there is the unique cusp form η^{24} where

$$
\eta(q) \equiv q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \quad (118)
$$

is the Dedekind η-function. (η does not vanish at ρ or 1.)

One can also construct a modular invariant function but it necessarily has a pole somewhere on the fundamental domain. The j–function defined by

$$
\begin{aligned}
j(q) &\equiv \frac{E_4^3}{\eta^{24}} = \frac{E_6^2}{\eta^{24}} + 1728 \\
&= q^{-1} + 744 + 196884q + \dots \quad (119)
\end{aligned}
$$

has a simple pole at $T = \infty$ and a triple zero at $T = \rho$. This function maps the fundamental domain of $SL(2, \mathbf{Z})$ onto the complex plane.

In general the derivative of a modular form is not a modular form since it does not satisfy eq. (116). An exception is the derivative $\partial_T^n F_{1-n}$ which transforms according to

$$
\partial_T^n F_{1-n} \to (icT + d)^{(n+1)} \partial_T^n F_{1-n} \quad (120)
$$

and thus is a modular form of weight $n + 1$.

B. Calabi-Yau manifolds

In this appendix we briefly recall a few facts about Calabi–Yau manifolds which we frequently use in the main text. (For a more extensive review see the lectures by R. Plesser or for example [1, 114].)

A Calabi–Yau manifold Y is a Ricci-flat Kähler manifold of vanishing first Chern-class. Its holonomy group is $SU(n)$ where n is the complex dimension of Y. The simplest Calabi–Yau manifolds are tori of complex dimension 1. For $n = 2$ all Calabi–Yau manifolds are topologically equivalent to the $K3$ surface (Kummers third surface), while for $n = 3$ one finds many topologically distinct Calabi–Yau threefolds. Such manifolds are of interest in string theory since they break some of the supersymmetries when a ten-dimensional string theory is compactified on Y. (See table 4,5.)

The massless modes of a string vacuum are directly related to the zero modes of the Laplace operator on Y. These zero modes are the non-trivial differential k-forms on Y and they are elements of the cohomology groups $H^k(Y)$. On a compact Kähler manifold one can decompose any k-form into a (p, q)-form with p holomorphic and q antiholomorphic differentials. Analogously, the associated cohomology groups decompose according to

$$
H^k(Y) = \oplus_{p+q=k} H^{p,q}(Y). \quad (121)
$$

The dimension of $H^{p,q}(Y)$ is called the Hodge number $h_{p,q}$ ($h_{p,q} = \dim H^{p,q}$); it is symmetric under the exchange of p and q, i.e. $h_{p,q} = h_{q,p}$, and Poincaré duality identifies $h_{p,q} = h_{n-p,n-q}$. Finally, the Euler number is given by

$$
\chi = \sum_{p,q} (-1)^{p+q} h_{p,q} . \quad (122)
$$

For a $K3$ surface one finds for the independent Hodge numbers $h_{0,0} = h_{2,0} = h_{2,2} = 1$, $h_{1,0} = h_{2,1} = 0$ and $h_{1,1} = 20$; hence eq. (122) implies $\chi(K3) = 24$. For a Calabi–Yau threefold Y the independent Hodge numbers obey $h_{0,0} = h_{3,0} = 1$, $h_{1,0} = h_{2,0} = 0$ while $h_{1,1}$ and $h_{1,2}$ are arbitrary. Thus eq. (122) implies $\chi(Y) = 2(h_{1,1} - h_{1,2})$. $h_{1,1}$ is the number of

Kähler moduli, which are nontrivial deformations of the metric and the antisymmetric tensor; $h_{1,2}$ is the number of moduli, which are nontrivial deformations of the complex structure. The moduli space is locally a direct product of the Kähler moduli space and the complex structure moduli space

$$\mathcal{M} = \mathcal{M}_{h_{1,1}} \otimes \mathcal{M}_{h_{1,2}} . \tag{123}$$

It is believed that most Calabi–Yau threefolds (if not all) have a mirror partner [122–124]. That is, for a given Calabi–Yau threefold Y with given $h_{1,1}(Y)$ and $h_{1,2}(Y)$ there exists a mirror manifold $\tilde{Y}$ with $h_{1,1}(\tilde{Y}) = h_{1,2}(Y)$ and $h_{1,2}(\tilde{Y}) = h_{1,1}(Y)$. (This implies in particular $\chi(Y) = -\chi(\tilde{Y})$.)

REFERENCES

1. M. Green, J. Schwarz and E. Witten, *Superstring Theory*, Cambridge University Press, 1987.
2. D. Lüst and S. Theisen, *Lectures on String Theory*, Springer Verlag, 1989.
3. M. Peskin, in *From the Planck Scale to the Weak Scale*, ed. H. Haber, World Scientific, 1987.
4. J. Polchinski, *What is String Theory*, Les Houches lectures 1994, hep-th/94111028.
5. D. Friedan, E. Martinec and S. Shenker, Nucl. Phys. B271 (1986) 93.
6. D.J. Gross and E. Witten, Nucl. Phys. B277 (1986) 1.
7. D.J. Gross and J.H. Sloan, Nucl. Phys. B291 (1987) 41.
8. For a review see, A. Giveon, M. Porrati and E. Rabinovici, Phys. Rep. 244 (1994), hep-th/9401139.
9. A. Schellekens and N. Warner, Phys. Lett. 177B (1986) 317.
10. A.A. Tseytlin, in *Superstrings '89*, ed. M. Green, R. Iengo, S. Randjbar-Daemi, E. Sezgin and A. Strominger, World Scientific, 1990.
11. C. Callan and L. Thorlacius, in *Particles, Strings and Supernovae*, ed. A. Jevicki and C.-I. Tan, World Scientific, 1989.
12. M. Dine and N. Seiberg, Phys. Rev. Lett. 55 (1985) 366.
13. V. Kaplunovsky, Phys. Rev. Lett. 55 (1985) 1036.
14. D. Lüst, S. Theisen and G. Zoupanos, Nucl. Phys. B296 (1988) 800.
15. L. Dixon, V. Kaplunovsky and J. Louis, Nucl. Phys. B329 (1990) 27.
16. E. Witten, Nucl. Phys. B460 (1996) 541, hep-th/9511030.
17. T. Banks, L. Dixon, D. Friedan and E. Martinec, Nucl. Phys. B299 (1988) 613.
18. T. Banks and L. Dixon, Nucl. Phys. B307 (1988) 93.
19. W. Lerche, A. Schellekens and N. Warner, Phys. Lett. B214 (1988) 41; Phys. Rep. 177 (1989) 1.
20. J. Wess and J. Bagger, *Supersymmetry and Supergravity*, Princeton University Press, 1983.
21. S. Ferrara and M. Villasante, Phys. Lett. 186B (1987) 85.
22. P. Binetruy, G. Girardi, R. Grimm and M. Müller, Phys. Lett. B195 (1987) 389.
23. S. Cecotti, S. Ferrara and M. Villasante, Int. J. Mod. Phys. A2 (1987) 1839.
24. P. Binetruy, Phys. Lett. B315 (1993) 80, hep-th/9305069.
25. R. D'Auria, S. Ferrara and M. Villasante, Class. Quant. Grav. 11 (1994) 481, hep-th/930612.
26. S. Kachru and C. Vafa, Nucl. Phys. B450 (1995) 69, hep-th/9505105.
27. V. Kaplunovsky, J. Louis and S. Theisen, Phys. Lett. B357 (1995) 71, hep-th/9506110.
28. A. Klemm, W. Lerche and P. Mayr, Phys. Lett. B357 (1995) 313, hep-th/9506112.
29. G. Lopes Cardoso and B. Ovrut, Nucl. Phys. B369 (1992) 351; Nucl. Phys. B392 (1993) 315, hep-th/9205009.
30. J.P. Derendinger, S. Ferrara, C. Kounnas and F. Zwirner, Nucl. Phys. B372 (1992) 145, Phys. Lett. B271 (1991) 30.
31. M.K. Gaillard and T. Taylor, Nucl. Phys. B381 (1992) 577.
32. P. Binetruy, G. Girardi and R. Grimm, Phys. Lett. B265 (1991) 111.
33. P. Adamietz, P. Binetruy, G. Girardi and R. Grimm, Nucl. Phys. B401 (1993) 257.
34. J.-P. Derendinger, F. Quevedo and M.

Quiros, Nucl. Phys. B428 (1994) 282, hep-th/9402007.

35. P. Binetruy, M.K. Gaillard and T.R. Taylor, Nucl. Phys. B455 (1995) 97, hep-th/9504143.

36. C.P. Burgess, J.P. Derendinger, F. Quevedo and M. Quiros, Phys. Lett. B348 (1995) 428, hep-th/9501065 and hep-th/9505171.

37. I. Gaida and D. Lüst, Phys. Lett. B367 (1996) 104, hep-th/9510022.

38. E. Cremmer, S. Ferrara, L. Girardello and A. Van Proeyen, Nucl. Phys. B212 (1983) 413.

39. S. Cecotti, S. Ferrara, L. Girardello and M. Porrati, Phys. Lett. 164B (1985) 46.

40. M. Grisaru, M. Roček and W. Siegel, Nucl. Phys. B159 (1979) 429.

41. M. Dine and N. Seiberg, Phys. Rev. Lett. 57 (1986) 2625.

42. M.A. Shifman and A.I. Vainshtein, Nucl. Phys. B277 (1986) 456 and Nucl. Phys. B359 (1991) 571.

43. H.P. Nilles, Phys. Lett. 180B (1986) 240.

44. V. Kaplunovsky and J. Louis, Nucl. Phys. B422 (1994) 57, hep-th/9402005.

45. M. Dine and Y. Shirman, Phys. Rev. D50 (1994) 5389.

46. J. Louis, in *Particles, Strings and Cosmology*, ed. P. Nath und S. Reucroft, World Scientific, 1992.

47. V. Kaplunovsky and J. Louis, Nucl. Phys. B444 (1995) 191, hep-th/9502077.

48. P. West, Phys. Lett. B258 (1991) 375 and Phys. Lett. B261 (1991) 396.

49. I. Jack, D.R.T. Jones and P. West, Phys. Lett. B258 (1991) 382.

50. D.C. Dunbar, I. Jack and D.R.T. Jones, Phys. Lett. B261 (1991) 62.

51. N. Seiberg, Phys. Lett B318 (1993) 469.

52. E. Poppitz and L. Randall, hep-th/9608157.

53. L. Dixon, V. Kaplunovsky and J. Louis, Nucl. Phys. B355 (1991) 649.

54. V. Novikov, M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, Nucl. Phys. B229 (1983) 381.

55. V. Kaplunovsky, Nucl. Phys. B307 (1988) 145, hep-th/9205070.

56. P. Mayr and S. Stieberger, Nucl. Phys. B412 (1994) 502, hep-th/9304055.

57. I. Antoniadis, K.S. Narain and T.R. Taylor, Phys. Lett. B267 (1991) 37.

58. I. Antoniadis, E. Gava and K. Narain, Phys. Lett. B283 (1992) 209, hep-th/9203071 and Nucl. Phys. B383 (1992) 93, hep-th/9204030.

59. P. Mayr and S. Stieberger, Nucl. Phys. B407 (1993) 725, hep-th/9303017.

60. E. Kiritsis and C. Kounnas, Nucl. Phys. B442 (1995) 472, hep-th/9501020 and Nucl. Phys. Proc. Suppl. 45BC (1996) 207, hep-th/9509017.

61. L. Dixon, J.A. Harvey, C. Vafa and E. Witten, Nucl. Phys. B261 (1985) 678 and Nucl. Phys. B274 (1986).

62. E. Witten, Phys. Lett. 155B (1985) 151.

63. M. Cvetič, J. Louis and B. Ovrut, Phys. Lett. B206 (1988) 227.

64. S. Ferrara and M. Porrati, Phys. Lett. B216 (1989) 289.

65. L. Ibáñez and D. Lüst, Nucl. Phys. B382 (1992) 305.

66. R. Dijkgraaf, E. Verlinde and H. Verlinde, in *Perspectives in String Theory*, ed. P. Di Vecchia and J.L. Perterson, World Scientific, 1988.

67. M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, Nucl. Phys. B405 (1993) 279, hep-th/9302103 and Comm. Math. Phys. 165 (1994) 311, hep-th/9309140.

68. P. Candelas, X. De la Ossa, A. Font, S. Katz and D. Morrison, Nucl. Phys. B416 (1994) 481, hep-th/9308083.

69. P. Candelas , A. Font, S. Katz and D. Morrison, Nucl. Phys. B429 (1994) 626, hep-th/9403187.

70. S. Hosono, A. Klemm, S. Theisen and S.T. Yau, Nucl. Phys. B433 (1995) 501, hep-th/9406055.

71. P. Berglund, S. Katz and A. Klemm, Nucl. Phys. B456 (1995) 153, hep-th/9506091.

72. N. Seiberg and E. Witten, Nucl. Phys. B426 (1994) 19, hep-th/9407087 and Nucl. Phys. B431 (1994) 484, hep-th/9408099.

73. S.J. Gates and R. Grimm, Z. Phys. C26 (1985) 621.

74. M. Sohnius, K. Stelle and P. West, Phys. Lett. B92 (1980) 123.

75. B. de Wit, V. Kaplunovsky, J. Louis and D. Lüst, Nucl. Phys. B451 (1995) 53, hep-

th/9504006.

76. P. Claus, B. de Wit, M. Faux, B. Kleijn, R. Siebelink and P. Termonia, Phys. Lett. B373 (1996) 81, hep-th/9512143.

77. A. Hindawi, B. Ovrut and D. Waldram, hep-th/9609016.

78. B. de Wit and J.W. van Holten, Nucl. Phys. B155 (1979) 530.

79. J. Bagger and E. Witten, Nucl. Phys. B222 (1983) 1.

80. B. de Wit and A. Van Proeyen, Nucl. Phys. B245 (1984) 89.

81. B. de Wit, P.G. Lauwers and A. Van Proeyen, Nucl. Phys. B255 (1985) 569.

82. L. Castellani, R. D'Auria and S. Ferrara, Phys. Lett. B 241 (1990) 57 and Class. Quant. Grav. 7 (1990) 1767.

83. R. D'Auria, S. Ferrara and P. Fré, Nucl. Phys. B359 (1991) 705.

84. L. Andrianopoli, M. Bertollini, A. Ceresole, R. D'Auria, S. Ferrara and P. Fré, hep-th/9603004.

85. L. Andrianopoli, M. Bertollini, A. Ceresole, R. D'Auria, S. Ferrara, P. Fré and T. Magri, hep-th/9605032.

86. A. Van Proeyen, hep-th/9512139.

87. A. Strominger, Comm. Math. Phys. 133 (1990) 163.

88. A. Ceresole, R. D'Auria, S. Ferrara, W. Lerche and J. Louis, Int. J. Mod. Phys. A8 (1993) 79, hep-th/9204035.

89. B. Craps, F. Roose, W. Troost and A. Van Proyen, hep-th/960673.

90. E. Cremmer, C. Kounnas, A. Van Proeyen, J.P. Derendinger, S. Ferrara, B. de Wit and L. Girardello, Nucl. Phys. B250 (1985) 385.

91. I. Antoniadis, S. Ferrara, E. Gava, K.S. Narain and T.R. Taylor, Nucl. Phys. B447 (1995) 35, hep-th/9504034.

92. A. Ceresole, R. D'Auria and S. Ferrara, Phys. Lett. B339 (1994) 71, hep-th/9408036.

93. A. Ceresole, R. D'Auria, S. Ferrara and A. Van Proeyen, Nucl. Phys. B444 (1995) 92, hep-th/9502072.

94. A. Strominger, Nucl. Phys. B451 (1995) 96, hep-th/9504090.

95. S. Ferrara and A. Van Proeyen, Class. Quant. Grav. 6 (1989) 243.

96. N. Seiberg, Phys. Lett. 206B (1988) 75.

97. J.A. Harvey and G. Moore, Nucl. Phys. B463 (1996) 315, hep-th/9510182.

98. T. Kawai, Phys. Lett. B372 (1996) 59, hep-th/9512046 and hep-th/9607078.

99. E. Kiritsis, C. Kounnas, P.M. Petropoulos and J. Rizos, hep-th/9608034.

100. R. Dijkgraaf, G. Moore, E. Verlinde and H. Verlinde, hep-th/9608096.

101. M. Henningson and G. Moore, Nucl. Phys. B472 (1996) 518, hep-th/9602154; hep-th/9608145.

102. G. Lopes Cardoso, G. Curio and D. Lüst, hep-th/9608154.

103. J.A. Harvey and G. Moore, hep-th/9609017.

104. P. Mayr and S. Stieberger, Phys. Lett. B 355 (1995) 107, hep-th/9504129.

105. I. Antoniadis and H. Partouche, Nucl. Phys. B460 (1996) 470, hep-th/9509009.

106. I. Antoniadis, E. Gava, K.S. Narain and T.R. Taylor, Nucl. Phys. B413 (1994) 162, hep-th/9307158 and Nucl. Phys. B476 (1996) 133, hep-th/9604077.

107. B. de Wit, Nucl. Phys. B Proc. Suppl. 49 (1996), hep-th/9602060.

108. I. Antoniadis, E. Gava, K.S. Narain and T.R. Taylor, Nucl. Phys. B455 (1995) 109, hep-th/9507115.

109. G. Curio, Phys. Lett. B366 (1996) 131, hep-th/9509042 and Phys. Lett. B368 (1996) 78, hep-th/9509146.

110. G. Lopes Cardoso, G. Curio, D. Lüst, T. Mohaupt and S. J. Rey, Nucl. Phys. B464 (1996) 18, hep-th/9512129.

111. B. de Wit, G. Lopes Cardoso, D. Lüst, T. Mohaupt and S. J. Rey, hep-th/9607184.

112. L. Dixon, V. Kaplunovsky and C. Vafa, Nucl. Phys. B294 (1987) 43.

113. C. Vafa and E. Witten, hep-th/9507050.

114. S. Hosono, A. Klemm and S. Theisen, in *Integrable Models and Strings*, Springer Lecture Notes in Physics 436, ed. A. Alekseev et al., hep-th/9403096.

115. N. Seiberg, Nucl. Phys. B303 (1988) 286.

116. S. Cecotti, S. Ferrara and L. Girardello, Int. J. Mod. Phys. A4 (1989) 2457.

117. S. Cecotti, Comm. Math. Phys. 124 (1989) 23.

118. S. Ferrara and S. Sabharwal, Nucl. Phys. B332 (1990) 317.

119. B. de Wit and A. Van Proeyen, Phys. Lett. B252 (1990) 221; Comm. Math. Phys. 149 (1992) 307, hep-th/9112027; Int. J. Mod. Phys. D3 (1994) 31, hep-th/9310067.

120. N. Seiberg and S. Shenker, hep-th/9608086.

121. L. Andrianopoli, R. D'Auria, S. Ferrara, P. Fré and M. Trigiante, hep-th/9611014.

122. L. Dixon, in *Superstrings, Unified Theories, and Cosmology* ed. G. Furlan et al., World Scientific (1988).

123. P. Candelas, M. Lynker and R. Schimmrigk, Nucl. Phys. B341 (1990) 383.

124. B. Greene and R. Plesser, Nucl. Phys. B338 (1990) 15.

125. P. Candelas, X. De la Ossa, P. Green and L. Parkes , Nucl. Phys. B359 (1991) 21, hep-th/9308083.

126. D. Morrison, hep-th/9111025.

127. A. Font, Nucl. Phys. B391 (1993) 358, hep-th/9203084.

128. A. Klemm and S. Theisen, Nucl. Phys. B389 (1993) 153, hep-th/9205041.

129. S. Hosono, A. Klemm, S. Theisen and S.T. Yau, Comm. Math. Phys. 167 (1995) 301, hep-th/9308122.

130. P. Berglund and S. Katz, Nucl. Phys. B420 (1994) 289, hep-th/9311014.

131. S. Ferrara, J.A. Harvey, A. Strominger and C. Vafa, Phys. Lett. B361 (1995) 59, hep-th/9505162.

132. P.S. Aspinwall, Phys. Lett. B357 (1995) 329, hep-th/9507012 ; Phys. Lett. B371 (1996) 231, hep-th/9511171.

133. E. Witten, hep-th/9507121.

134. G. Lopes Cardoso, G. Curio, D. Lüst and T. Mohaupt, hep-th/9603108.

135. G. Aldazabal, A. Font, L.E. Ibañéz and F. Quevedo, Nucl. Phys. B461 (1996) 85, hep-th/9510093 and Phys. Lett. B380 (1996) 33, hep-th/9602097.

136. D. Morrison, hep-th/9512016.

137. P. Candelas and A. Font, hep-th/9603170.

138. P. Berglund, S. Katz, A. Klemm and P. Mayr, hep-th/9605154.

139. J. Louis, J. Sonnenschein, S. Theisen and S. Yankielowicz, hep-th/9606049.

140. D. Morrison and C. Vafa, Nucl. Phys. B473 (1996) 74, hep-th/9602114 and Nucl. Phys. B476 (1996) 437, hep-th/9603161.

141. P.S. Aspinwall and M. Gross, Phys. Lett. B382 (1996) 81, hep-th/9602118.

142. A. Klemm, W. Lerche, P. Mayr, C. Vafa and N. Warner, hep-th/9604034.

143. S. Kachru, A. Klemm, W. Lerche, P. Mayr and C. Vafa, Nucl. Phys. B459 (1996) 537, hep-th/9508155.

144. G. Lopes Cardoso, D. Lüst and T. Mohaupt, Nucl. Phys. B455 (1995) 131, hep-th/9507113.

145. P.S. Aspinwall and J. Louis, Phys. Lett. B369 (1996) 233, hep-th/9510234.

146. B. Hunt and R. Schimmrigk, Phys. Lett. B381(1996) 427, hep-th/9512138.

147. P.S. Aspinwall and D.R. Morrison, hep-th/9404151.

148. B.R. Greene, D.R. Morrison and A. Strominger, Nucl. Phys. B451 (1995) 109, hep-th/9504145.

149. S. Katz, D. Morrison and R. Plesser, Nucl. Phys. B477 (1996) 105, hep-th/9601108.

150. A. Klemm and P. Mayr, hep-th/9601014.

151. T.M. Apostol, *Modular Functions and Dirichlet Series in Number Theory*, Springer-Verlag, 1989.

152. J.P. Serre, *A Course in Arithmetic*, Springer Verlag, 1973.

153. B. Schoeneberg, *Elliptic Modular Functions*, Springer-Verlag, 1974.